Shape of the Unitary Triangle
and Phase Conventions of the CKM Matrix

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Abstract

A shape of the unitary triangle versus a CP violating parameter δ depends on the phase conventions of the CKM matrix, although relations among observable quantities are independent of the phase conventions. In order to seek for a clue to the quark mass matrix structure and the origin of the CP violation, the dependence of the unitary triangle shape on the parameter δ is systematically investigated.

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1 Introduction

Usually, it is taken that any phase conventions of the Cabibbo-Kobayashi-Maskawa (CKM) [1, 2] matrix are equivalent to each other because of the rephasing invariance. This is true, as far as the observable quantities are concerned. However, if we want to put some ansatz on the quark mass matrices and/or the CKM matrix, what phase convention we adopt becomes an important concern to us. For example, by noticing that predictions based on the maximal CP violation hypothesis [3] depend on the phase convention, the author [4] has recently pointed out that we can obtain successful predictions on the unitary triangle only when we adopt the original Kobayashi-Maskawa (KM) [1] phase convention and the Fritzsch-Xing [5] phase convention. If we put the ansatz on the standard phase convention [6] of the CKM matrix, we will obtain wrong results on the unitary triangle. For experimental studies, what convention we adopt is not important, but, for model-building of the quark and lepton mass matrices, it is a big concern. In the present paper, in order to look for a clue to the origin of the CP violating phase δ (what elements in the quark mass matrices contain the CP violating phase δ and how the magnitude of δ is), we will systematically investigate whole phase conventions of the CKM matrix, comparing with the present experimental data of the unitary triangle.

Recent remarkable progress of the experimental B physics [7] has put the shape of the unitary triangle within our reach. The world average value of the angle β [8] which has been obtained from B_d decays is

$$\sin 2\beta = 0.736 \pm 0.049 \quad \left(\beta = 23.7^{\circ}{}^{+2.2^{\circ}}_{-2.0^{\circ}}\right) , \tag{1.1}$$

and the best fit [8] for the CKM matrix V also gives

$$\gamma = 60^{\circ} \pm 14^{\circ} \ , \quad \beta = 23.4^{\circ} \pm 2^{\circ} \ , \tag{1.2}$$

where the angles α, β and γ are defined by

$$\alpha \equiv \phi_2 = \mathrm{Arg}\left[-\frac{V_{31}V_{33}^*}{V_{11}V_{13}^*}\right], \quad \beta \equiv \phi_1 = \mathrm{Arg}\left[-\frac{V_{21}V_{23}^*}{V_{31}V_{33}^*}\right], \quad \gamma \equiv \phi_3 = \mathrm{Arg}\left[-\frac{V_{11}V_{13}^*}{V_{21}V_{23}^*}\right]. \tag{1.3}$$

Also we know the observed values [8] of the magnitudes $|V_{ij}|$ of the CKM matrix elements:

$$|V_{us}| = 0.2200 \pm 0.0026, \quad |V_{cb}| = 0.0413 \pm 0.0015, \quad |V_{ub}| = 0.00367 \pm 0.00047, \tag{1.4}$$

$$\mathrm{Re}V_{td} = 0.0067 \pm 0.0008, \quad \mathrm{Im}V_{td} = -0.0031 \pm 0.0004. \tag{1.5}$$

Thus, nowadays, we have almost known the shape of the unitary triangle $V_{ud}^*V_{ub} + V_{cd}^*V_{cb} + V_{td}^*V_{tb} = 0$. We are interested what logic can give the observed magnitude of the CP violation.

There are, in general, 9 independent phase conventions [9] of the CKM matrix. In the present paper, we define the expressions of the CKM matrix V as

$$V = V(i,k) \equiv R_i^T P_j R_j R_k \qquad (i \neq j \neq k), \tag{1.6}$$

where

$$R_1(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix}, \quad R_2(\theta) = \begin{pmatrix} c & 0 & s \\ 0 & 1 & 0 \\ -s & 0 & c \end{pmatrix}, \quad R_3(\theta) = \begin{pmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{1.7}$$

($s = \sin\theta$ and $c = \cos\theta$) and

$$P_1 = \mathrm{diag}(e^{i\delta},\ 1,\ 1), \quad P_2 = \mathrm{diag}(1,\ e^{i\delta},\ 1), \quad P_3 = \mathrm{diag}(1,\ 1,\ e^{i\delta}). \tag{1.8}$$

The expressions $V(1,3)$, $V(1,1)$ and $V(3,3)$ correspond to the standard [6], original KM [2] and Fritzsch-Xing [5] phase conventions, respectively.

By the way, the CKM matrix structure (1.6) is related to a quark mass matrix model under the following specific assumption: We assume that the phase factors in the quark mass matrices M_f $(f = u, d)$ can be factorized by the phase matrices P_f as

$$M_f = P_{fL}^\dagger \widetilde{M}_f P_{fR}\ , \tag{1.9}$$

where P_f are phase matrices and $\widetilde{M}_f$ are real matrices. (This is possible for a mass matrix which has specific zero-textures.) The real matrices $\widetilde{M}_f$ are diagonalized by rotation (orthogonal) matrices R_f as

$$R_f^\dagger \widetilde{M}_f R_f = D_f \equiv \mathrm{diag}(m_{f1},\ m_{f2},\ m_{f3}), \tag{1.10}$$

[for simplicity, we have assumed that M_f are Hermitian (or symmetric) matrix, i.e. $P_{fR} = P_{fL}$ (or $P_{fR} = P_{fL}^*$)], so that the CKM matrix V is given by

$$V = R_u^T P R_d\ , \tag{1.11}$$

where $P = P_{uL}^\dagger P_{dL}$. The quark masses m_{fi} are only determined by $\widetilde{M}_f$. In other words, the rotation parameters are given only in terms of the quark mass ratios, and independent of the CP violating phases. In such a scenario, the CP violation parameter δ can be adjusted without changing the quark mass values. In the present paper, by fixing the rotation matrices R_u and R_d (i.e. by fixing the quark masses), we tacitly assume that the CP violation is described only by the adjustable parameter δ. Then, the expression of the law of the CP violation depends on the phase conventions of the CKM matrix.

For example, the phase convention $V(2,3)$

$$V(2,3) = R_2^T(\theta_{13}^u) P_1(\delta) R_1(\theta_{23}) R_3(\theta_{12}^d), \tag{1.12}$$

suggests the quark mass matrix structures

$$\begin{aligned} \widetilde{M}_u &= R_1(\theta_{23}^u) R_2(\theta_{13}^u) D_u R_2^T(\theta_{13}^u) R_1^T(\theta_{23}^u) \ , \\ \widetilde{M}_d &= R_1(\theta_{23}^d) R_3(\theta_{12}^d) D_d R_3^T(\theta_{12}^d) R_1^T(\theta_{23}^d) \ , \end{aligned} \tag{1.13}$$

with $\theta_{23} = \theta_{23}^d - \theta_{23}^u$. Therefore, in order to seek for a clue to the quark mass matrix structure, we interest in the relations of the phase conventions (1.6) to the observed unitary triangle shape.

2 Rephasing invariant quantity J versus δ

Of the three unitary triangles $\triangle^{(ij)}$ $[(ij) = (12),\ (23),\ (31)]$ which denote the unitary conditions

$$\sum_k V_{ki}^* V_{kj} = \delta_{ij}, \tag{2.1}$$

we usually discuss the triangle $\triangle^{(31)}$, i.e.

$$V_{ud}^* V_{ub} + V_{cd}^* V_{cb} + V_{td}^* V_{tb} = 0, \tag{2.2}$$

because the triangle $\triangle^{(31)}$ is the most useful one for the experimental studies.

The rephasing invariant quantity [10] J is given by

$$J = \frac{|V_{i1}||V_{i2}||V_{i3}||V_{1k}||V_{2k}||V_{3k}|}{(1-|V_{ik}|^2)|V_{ik}|} \sin\delta, \tag{2.3}$$

in the phase convention $V(i,k)$, where the CP violating phase δ has been defined by Eq. (1.7). Note that the 5 quantities (not 6 quantities) $|V_{i1}|$, $|V_{i2}|$, $|V_{i3}|$, $|V_{1k}|$, $|V_{2k}|$ and $|V_{3k}|$ in the expression $V(i,k)$ are independent of the phase parameter δ. (In other words, only the remaining 4 quantities are dependent of δ.) Therefore, the rephasing invariant quantity J is dependent on the parameter δ only through the factor $\sin\delta$. A "maximal CP violation" means a maximal J, so that it means a maximal $\sin\delta$. Thus, the maximal CP violation hypothesis depends on the phase conventions.

From the expression (2.3), for the observed fact $1 \gg |V_{us}|^2 \simeq |V_{cd}|^2 \gg |V_{cb}|^2 \simeq |V_{ts}|^2 \gg |V_{ub}|^2$, the rephasing invariant quantity J is classified in the following four types:

$$\begin{array}{ll} (A): & J \simeq |V_{ub}||V_{td}| \sin\delta, \\ (B): & J \simeq |V_{us}||V_{cb}||V_{ub}| \sin\delta, \\ (C): & J \simeq |V_{us}||V_{cb}||V_{td}| \sin\delta, \\ (D): & J \simeq |V_{cb}|^2 \sin\delta. \end{array} \tag{2.4}$$

The corresponding phase conventions $V(i,k)$ are listed in Table 1.

The present experimental values (1.2) suggest $\alpha \simeq 90^\circ$. Since only the cases $V(1,1)$ and $V(3,3)$ can give $\delta \simeq \alpha$ as seen in Table 1, the "maximal CP violation hypothesis" (i.e. maximal $\sin\delta$ hypothesis) can give successful results only for the cases $V(1,1)$ and $(3,3)$ [4].

3 Angles ϕ_i versus δ

In the present section, we systematically investigate the relations between the angles ϕ_ℓ ($\ell = 1,2,3$) and the CP violating phase δ for each case $V(i,k)$.

The angles $(\phi_1, \phi_2, \phi_3) \equiv (\beta, \alpha, \gamma)$ on the unitary triangle $\triangle^{(31)}$ are given by the sine rule

$$\frac{r_1}{\sin\phi_1} = \frac{r_2}{\sin\phi_2} = \frac{r_3}{\sin\phi_3} = 2R, \tag{3.1}$$

where R is the radius of the circumscribed circle of the triangle $\triangle^{(31)}$, and r_i are defined by

$$r_1 = |V_{13}||V_{11}|, \quad r_2 = |V_{23}||V_{21}|, \quad r_3 = |V_{33}||V_{31}|. \tag{3.2}$$

Then, the quantity J is rewritten as follows:

$$J = 2 r_m r_n \sin\phi_\ell = \frac{1}{R} r_\ell r_m r_n = \frac{1}{R} |V_{11}||V_{21}||V_{31}||V_{13}||V_{23}||V_{33}|, \tag{3.3}$$

where (ℓ, m, n) is a cyclic permutation of (1,2,3). From Eqs. (2.3), (3.1) and (3.3), the angles ϕ_ℓ are given by the formula

$$\sin\phi_\ell = \frac{|V_{i1}||V_{i2}||V_{i3}||V_{1k}||V_{2k}||V_{3k}| \sin\delta}{|V_{m1}||V_{m3}||V_{n1}||V_{n3}|(1-|V_{ik}|^2)|V_{ik}|}. \tag{3.4}$$

Of the three sides in the expression $V(i,k)$, only one side r_i is always independent of the phase parameter δ. And, of the three angle ϕ_i, only one (we express it with ϕ_ℓ), except for the case $V(2,2)$, is approximately equal to the phase parameter δ. In Table 1, we also list the side r_i which is independent of δ and the angle ϕ_ℓ which is approximately equal to δ.

The relations between ϕ_i ($i = 1,2,3$) and δ are illustrated in Figs. 1–8. The curves have been evaluated by using the explicit expression (1.6) (not by using the formula (3.4)). In general, there are five $|V_{ij}|$ which are independent of the phase parameter δ. For the cases that $|V_{us}|$, $|V_{cb}|$ and $|V_{ub}|$ are δ-independent V_{ij}, we have used the observed values (1.4) as the input values, i.e. $|V_{us}| = 0.22$, $|V_{cb}| = 0.0413$ and $|V_{ub}| = 0.00367$. When $|V_{us}|$ ($|V_{cb}|$) is δ-dependent, but $|V_{cd}|$ ($|V_{ts}|$) is δ-independent, we have, for convenience, used the input values $|V_{cd}| = 0.22$ ($|V_{ts}| = 0.0413$). When $|V_{ub}|$ is δ-dependent, but $|V_{td}|$ is δ-independent, we have, for convenience, used the input values $|V_{td}| = 0.0084$, which is a predicted value of $|V_{td}|$ in the case $V(1,1)$ with the maximal $\sin\delta$. However, for the case $V(2,2)$, since both $|V_{ub}|$ and $|V_{td}|$ are δ-dependent, so that we cannot use such an approximate substitute. As seen in Table 1, the case $V(2,2)$ needs a small value of δ compared with other cases, so that the case is not so interesting. We omit the case $V(2,2)$ from the present study.

As seen in Figs. 1–8, of the maximal values of the three $\sin\phi_i$ ($i = 1,2,3$), two can take $(\sin\phi_i)_{max} = 1$, while one (we express it with ϕ_s) always takes a smaller value than one, i.e. $(\sin\phi_s)_{max} < 1$. The angle ϕ_s with $(\sin\phi_s)_{max} < 1$ is ϕ_1 for the cases A and B, and is ϕ_3 for the case C. If we assume that nature chooses the value of the phase parameter δ such as $\sin\phi_s$ is maximal, as shown in Table 2, the cases $V(i,k)$ with $i \neq k$ can predict reasonable values of the angles ϕ_i ($i = 1,2,3$).

More straightforward ansatz is as follow: the value of $\sin\alpha$ has to take its maximal value $\sin\alpha = 1$. Then, all cases $V(i,k)$ can give reasonable values of the angles as seen in Table 2. However, this ansatz is merely other expression of the observed fact (1.2). In the maximal CP violation hypothesis, the hypothesis has been imposed on the CP violating phase parameter δ, which is not a directly observable quantity. Therefore, the hypothesis could choose specific phase conventions $V(1,1)$ and $V(3,3)$ (consequently, specific quark mass matrix structures) as experimentally favorable ones. In contrast to the maximal CP violation hypothesis, the ansatz for the directly observable quantities such as $(\sin\alpha)_{max} = 1$ cannot choose a specific phase convention $V(i,k)$ as a favorable one. It is unlikely that the ansatz $\sin\alpha = 1$ gives a clue to the origin of the CP violating phase in the quark mass matrices.

4 Radius of the circumscribed circle

When we see the unitary triangle from the geometrical point of view, we find that the triangle $\triangle^{(31)}$ has the plumpest shape compared with other triangles $\triangle^{(12)}$ and $\triangle^{(23)}$, so that the triangle $\triangle^{(31)}$ has the shortest radius R_{min} of the circumscribed circle compared with the other cases $\triangle^{(12)}$ and $\triangle^{(23)}$. Therefore, let us put the following assumption: the phase parameter δ takes the value so that the radius of the circumscribed circle $R(\delta)$ takes its minimum value. The radius $R(\delta)$ is given by the sine rule (3.1). Note that the side r_i in the expression $V(i,k)$ is independent of the parameter δ. Therefore, the minimum of the radius $R(\delta)$ means the maximum of $\sin\phi_i(\delta)$ in the phase convention $V(i,k)$. In Table 3, we list values of (ϕ_1, ϕ_2, ϕ_3) at $\delta = \delta_0$ at which $\sin\phi_i$ takes its maximal value. As seen in Table 3, all cases except for $V(1,1)$ and $V(3,3)$ (and also $V(2,2)$) can give favorable predictions. Therefore, this ansatz is also not useful to select a specific $V(i,k)$.

If we put further strong constraint that the phase parameter δ takes own value so that

$\sin\phi_i(\delta)$ takes its maximal value $\sin\phi_i = 1$, then, we find that the possible candidates are only two: $V(2,3)$ and $V(2,1)$. (The other cases cannot take the value $\sin\phi_i = 1$ under the observed values (1.4) of $|V_{us}|$, $|V_{cb}|$ and $|V_{ub}|$.) When we take account of the forms of the quark mass matrices (M_u, M_d) which are suggested by Eq. (1.11) from a specific phase convention $V(i,k)$, we especially interest in the phase convention $V(2,3)$. The phase convention (1.12) suggests the quark mass matrix structure (1.13). It is well known that if we require the zero-texture $(M_d)_{11} = 0$ for the down-quark mass matrix M_d, we can obtain the successful prediction for $|V_{us}|$ [11]

$$|V_{us}| \simeq \sqrt{\frac{m_d}{m_s}} = 0.22. \tag{4.1}$$

From the point of view of M_u-M_d correspondence, if we also apply the zero-texture hypothesis to the up-quark mass matrix M_u, we obtain

$$|V_{ub}| \simeq s_{13}^u \simeq \sqrt{\frac{m_u}{m_t}} = 0.0036, \tag{4.2}$$

from $(M_u)_{11} = (m_{u3} - m_{u1}) c_{13}^u s_{13}^u c_{23}^u$, where we have used the quark mass values [12] at $\mu = m_Z$. The prediction is in excellent agreement with the observed value (1.4). (If we put $(M_u)_{11} = 0$ on the mass matrix M_u which is suggested from the phase convention $V(3,3)$, we will obtain $|V_{ub}/V_{cb}| \simeq \sqrt{m_u/m_c} = 0.059$, which is in poor agreement with the observed value $|V_{ub}/V_{cb}| = 0.089^{+0.015}_{-0.014}$.) Therefore, from the phenomenological point of view, we are interested in the phase convention $V(2,3)$ rather than the phase convention $V(3,3)$.

5 Concluding remarks

In conclusion, we have investigated the dependence of the unitary triangle shape on the CP violating parameter δ which is dependent on the phase conventions of the CKM matrix. The phase conventions are, generally, classified into the 9 expressions $V(i,k)$, Eq. (1.6), which suggests the quark mass matrix structures (1.9) with Eq. (1.11). If we require that the angle α ($\equiv \phi_2$) takes $\sin\alpha = 1$, all cases can predict favorable values of (ϕ_1, ϕ_2, ϕ_3) as seen in Table 2.

However, we want to select a specific phase convention $V(i,k)$ in order to seek for a clue to the quark mass matrix structure and the origin of the CP violation. Then, the most naive and simplest hypothesis is the well-known "maximal CP violation hypothesis", which means the requirement $\sin\delta = 1$. The ansatz selects the cases $V(1,1)$ and $V(3,3)$. The relations between $V(3,3)$ and the quark mass matrices (M_u, M_d) have already discussed in Refs. [5, 13].

Another selection rule is a minimal circumscribed circle hypothesis, which requires a maximal value of $\sin\phi_i$ in the phase convention $V(i,k)$. The hypothesis selects all cases except for $V(i,i)$ $(i = 1,2,3)$ as favorable ones. Only when we put a stronger constraint $\sin\phi_i = 1$, we can selects cases $V(2,3)$ and $V(2,1)$. (In other cases, $\sin\phi_i$ cannot take $\sin\phi_i = 1$ under the observed values (1.4) of $|V_{us}|$, $|V_{cb}|$ and $|V_{ub}|$.) We are interested in the case $V(2,3)$ because the suggested quark mass matrices predict successful relations $|V_{ub}| \simeq \sqrt{m_u/m_t}$ and $|V_{us}| \simeq \sqrt{m_d/m_s}$ under the simple texture-zero hypotheses $(M_u)_{11} = 0$ and $(M_d)_{11} = 0$, respectively.

Although, in the present paper, we did not discuss the neutrino mixing matrix [14] $U = U_{eL}^\dagger U_{\nu L}$, where $U_{eL}^\dagger M_e U_{eR} = D_e$ and $U_{\nu L}^\dagger M_\nu U_{\nu L}^* = D_\nu$, the expressions $V(i,k)$ will also be useful for studies of the neutrino mixings. If we obtain data of CP violation in the lepton sector in the near future, we can select a favorable expression $V(i,k)$ for the mixing matrix U, and thereby, we will be able to get a clue for investigating structures of M_e and M_ν individually.

References

[1] N. Cabibbo, Phys. Rev. Lett. **10**, 531 (1963).

[2] M. Kobayashi and T. Maskawa, Prog. Theor. Phys. **49**, 652 (1973).

[3] D. Hochberg and R. Sachs, Phys. Rev. **D27**, 606 (1983); B. Stech, Phys. Lett. **B130**, 189 (1983); L. Wolfenstein, Phys. Lett. **B144**, 425 (1984); M. Shin, Phys. Lett. **B145**, 285 (1984); H. Fritzsch, Phys. Rev. **D32**, 3058 (1985); M. Gronau and J. Schechter, Phys. Rev. Lett. **54**, 385 (1985); M. Gronau, R. Jonson and J. Schechter, Phys. Rev. Lett. **54**, 2176 (1985); F. J. Botella and L. -L. Chau, Phys. Lett. **B168**, 97 (1984). And also see, C. Jarlskog, in *Introduction to CP Violation*, edited by C. Jarlskog (World Scientific, Singapore, 1989), p.3, and references therein.

[4] Y. Koide, Phys. Lett. **B607**, 123 (2005).

[5] H. Fritzsch and Z. -z. Xing, Phys. Lett. **B413**, 396 (1997).

[6] L. -L. Chau and W. -Y. Keung, Phys. Rev. Lett. **53**, 1802 (1984); H. Fritzsch, Phys. Rev. **D32**, 3058 (1985); Phys. Lett. **166B**, 423 (1986); H. Harari and M. Leurer, Phys. Lett. **B181**, 123 (1986); H. Fritzsch and J. Plankl, Phys. Rev. **D35**, 1732 (1985); F. J. Botella and L. -L. Chao, Phys. Lett. **B168**, 97 (1986).

[7] For instance, B. Aubert *et al.*, Phys. Rev. Lett. **89**, 201802 (2002); K. Abe *et al*, Belle-CONF-0353, LP'03 (2003).

[8] S. Eidelman *et al.* (Particle Data Group), Phys. Lett. **B592**, 1 (2004) (URL: http://pdg.lbl.gov).

[9] H. Fritzsch and Z. -z. Xing, Phys. Rev. **D57**, 594 (1998).

[10] C. Jarlskog, Phys. Rev. Lett. **55**, 1839 (1985); O. W. Greenberg, Phys. Rev. **D32**, 1841 (1985); I. Dunietz, O. W. Greenberg and D.-d. Wu, Phys. Rev. Lett. **55**, 2935 (1985); C. Hamzaoui and A. Barroso, Phys. Rev. **D33**, 860 (1986).

[11] S. Weinberg, Ann. N.Y. Acad. Sci. **38**, 185 (1977); H. Fritzsch, Phys. Lett. **73B**, 317 (1978); Nucl. Phys. **B155**, 189 (1979); H. Georgi and D. V. Nanopoulos, *ibid.* **B155**, 52 (1979).

[12] H. Fusaoka and Y. Koide, Phys. Rev. **D57**, 3986 (1998).

[13] Z. -z. Xing, Phys. Rev. **D68**, 073008 (2003).

[14] Z. Maki, M. Nakagawa and S. Sakata, Prog. Theor. Phys. **28**, 870 (1962); B. Pontecorvo, Zh. Eksp. Theor. Fiz. **33**, 549 (1957); Sov. Phys. JETP **26**, 984 (1968).

Table 1 Classification of $V(i,k)$. The cases are classified under the approximation of $1 \gg |V_{us}|^2 \simeq |V_{cd}|^2 \gg |V_{cb}|^2 \simeq |V_{ts}|^2 \gg |V_{ub}|^2$. For the types of J, see Eq. (2.8) in the text.

Phase convention	Type of J	δ-independent r_i	$\delta \simeq \phi_\ell$
$V(1,1) = R_1^T P_2 R_2 R_1$	A	r_1	$\delta \simeq \phi_2$
$V(3,3) = R_3^T P_1 R_1 R_3$	A	r_3	$\delta \simeq \phi_2$
$V(1,2) = R_1^T P_3 R_3 R_2$	B	r_1	$\delta \simeq \phi_3$
$V(1,3) = R_1^T P_2 R_2 R_3$	B	r_1	$\delta \simeq \phi_3$
$V(2,3) = R_2^T P_1 R_1 R_3$	B	r_2	$\delta \simeq \phi_3$
$V(2,1) = R_2^T P_3 R_3 R_1$	C	r_2	$\delta \simeq \phi_1$
$V(3,1) = R_3^T P_2 R_2 R_1$	C	r_3	$\delta \simeq \phi_1$
$V(3,2) = R_3^T P_1 R_1 R_2$	C	r_3	$\delta \simeq \phi_1$
$V(2,2) = R_2^T P_1 R_1 R_2$	D	r_2	No simple relation

Table 2 Maximal $\sin\phi_s$ hypothesis.

		$(\sin\phi_s)_{max}$ (<1) at $\delta=\delta_0$					$(\sin\phi_2)_{max}=1$ at $\delta=\delta_0$		
Type	$V(i,k)$	s	ϕ_1	ϕ_2	ϕ_3	δ_0	ϕ_1	ϕ_3	δ_0
A	$V(1,1)$	$s=1$	25.4°	64.6°	90.0°	115.3°	23.2°	66.8°	90.0°
A	$V(3,3)$	$s=1$	23.2°	65.7°	91.1°	66.8°	21.4°	68.6°	91.1°
B	$V(1,2)$	$s=1$	22.8°	91.0°	66.2°	114.8°	22.8°	67.2°	113.8°
B	$V(1,3)$	$s=1$	23.2°	90.0°	66.8°	66.9°	23.2°	66.8°	66.9°
B	$V(2,3)$	$s=1$	23.2°	90.0°	66.8°	113.2°	23.2°	66.8°	113.2°
C	$V(2,1)$	$s=3$	22.5°	90.0°	67.5°	157.5°	22.5°	67.5°	157.5°
C	$V(3,1)$	$s=3$	25.7°	88.9°	65.4°	26.9°	24.6°	65.4°	25.7°
C	$V(3,2)$	$s=3$	25.6°	88.9°	65.5°	153.3°	24.5°	65.5°	154.4°

Table 3 Minimal circumscribed circle hypothesis. The hypothesis requires a maximal $\sin\phi_i$ in the phase convention $V(i,k)$. The underlined values are obtained by the maximal $\sin\phi_i$ requirement.

Type	$V(i,k)$	ϕ_1	ϕ_2	ϕ_3	δ_0
A	$V(1,1)$	$\underline{25.4^\circ}$	64.6°	90.0°	115.3°
A	$V(3,3)$	23.2°	66.8°	$\underline{90.0^\circ}$	67.8°
B	$V(1,2)$	$\underline{22.8^\circ}$	91.0°	66.2°	114.8°
B	$V(1,3)$	$\underline{23.2^\circ}$	90.0°	66.8°	66.9°
B	$V(2,3)$	23.2°	$\underline{90.0^\circ}$	66.8°	113.2°
C	$V(2,1)$	22.5°	$\underline{90.0^\circ}$	67.5°	157.5°
C	$V(3,1)$	25.7°	88.9°	$\underline{65.4^\circ}$	26.9°
C	$V(3,2)$	25.6°	88.9°	$\underline{65.5^\circ}$	153.3°

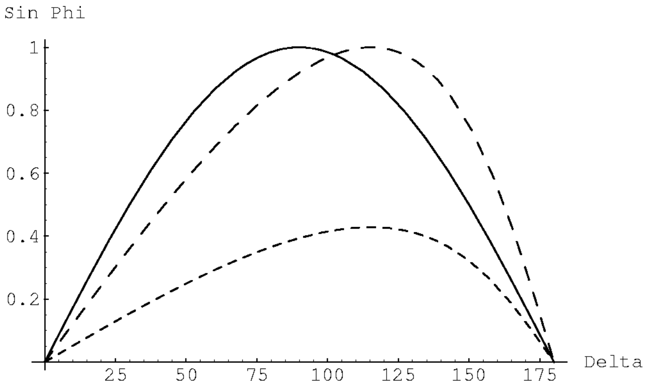

Figure 1: $\sin\phi_i$ $(i=1,2,3)$ versus δ in $V(1,1)$. The curves $\sin\alpha$, $\sin\beta$ and $\sin\gamma$ are denoted by a solid line, a dotted line and a dashed line, respectively.

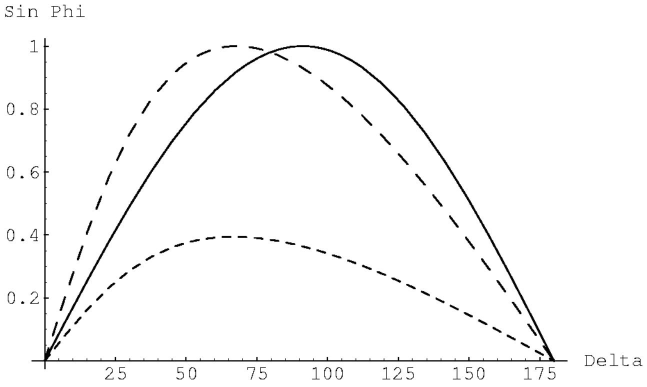

Figure 2: $\sin\phi_i$ $(i=1,2,3)$ versus δ in $V(3,3)$. The curves $\sin\alpha$, $\sin\beta$ and $\sin\gamma$ are denoted by a solid line, a dotted line and a dashed line, respectively.

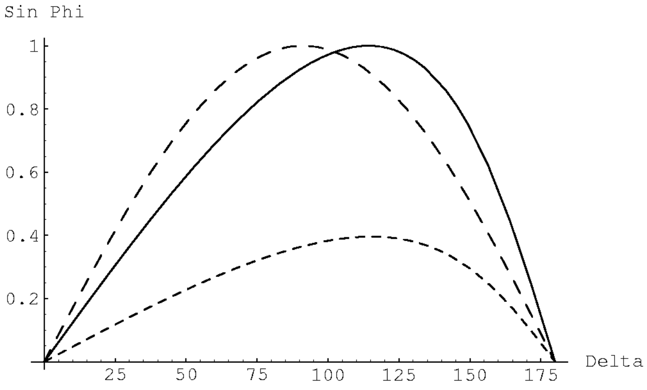

Figure 3: $\sin\phi_i$ $(i = 1, 2, 3)$ versus δ in $V(1, 2)$. The curves $\sin\alpha$, $\sin\beta$ and $\sin\gamma$ are denoted by a solid line, a dotted line and a dashed line, respectively.

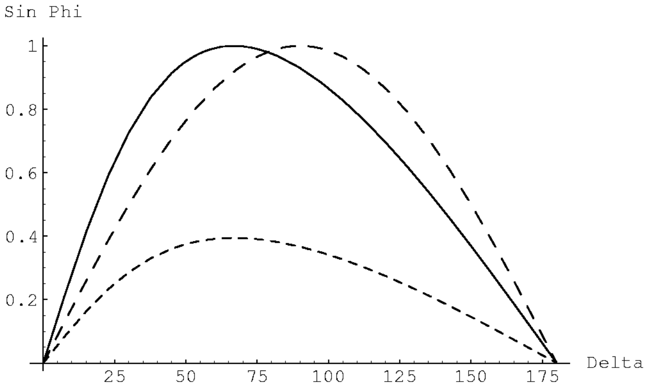

Figure 4: $\sin\phi_i$ $(i = 1, 2, 3)$ versus δ in $V(1, 3)$. The curves $\sin\alpha$, $\sin\beta$ and $\sin\gamma$ are denoted by a solid line, a dotted line and a dashed line, respectively.

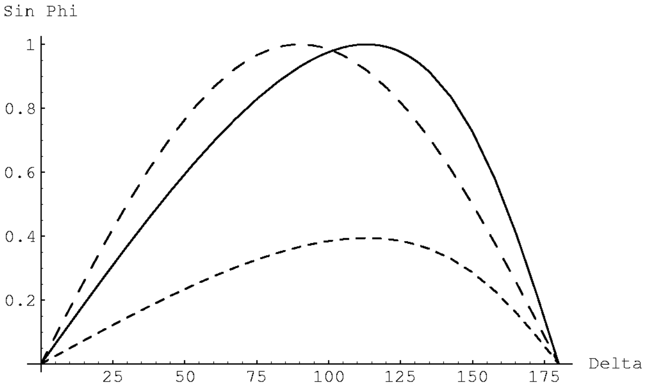

Figure 5: $\sin\phi_i$ $(i = 1, 2, 3)$ versus δ in $V(2, 3)$. The curves $\sin\alpha$, $\sin\beta$ and $\sin\gamma$ are denoted by a solid line, a dotted line and a dashed line, respectively.

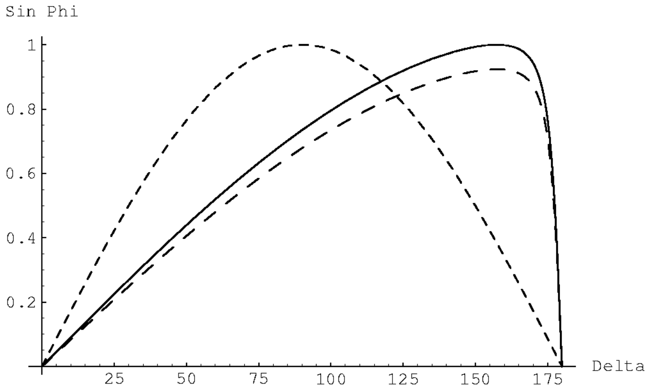

Figure 6: $\sin\phi_i$ $(i = 1, 2, 3)$ versus δ in $V(2, 1)$. The curves $\sin\alpha$, $\sin\beta$ and $\sin\gamma$ are denoted by a solid line, a dotted line and a dashed line, respectively.

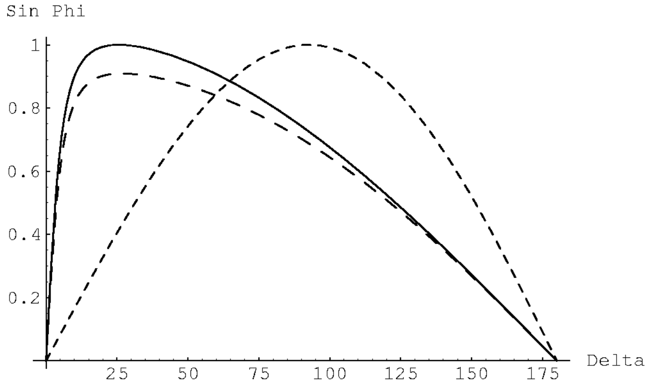

Figure 7: $\sin\phi_i$ $(i = 1, 2, 3)$ versus δ in $V(3, 1)$. The curves $\sin\alpha$, $\sin\beta$ and $\sin\gamma$ are denoted by a solid line, a dotted line and a dashed line, respectively.

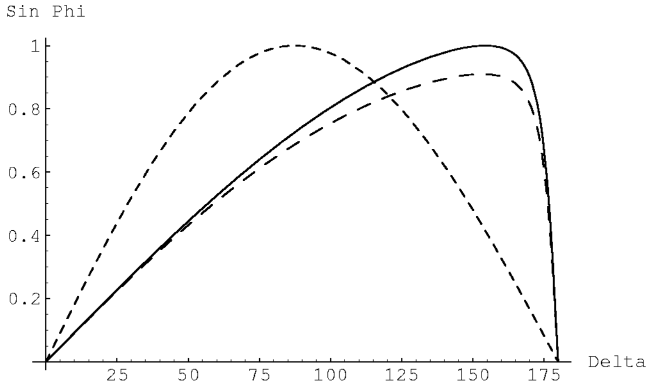

Figure 8: $\sin\phi_i$ $(i = 1, 2, 3)$ versus δ in $V(3, 2)$. The curves $\sin\alpha$, $\sin\beta$ and $\sin\gamma$ are denoted by a solid line, a dotted line and a dashed line, respectively.