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# Mass Matrix and its Related Formulae

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## Preface

For investigating a realistic model of the quarks and leptons, the phenomenological studies of the masses and mixings will provide a promising clue. Then, the investigation should simultaneously be done both for masses and mixings, so that we usually take a mass matrix approach. Many researchers have been reported along this line, and also, many useful formulae for the mass matrix calculations have found. With the progress of the recent neutrino physics, the concern to the mass matrix model-builders seems to be moving from quark mass matrix models to neutrino mass matrix models. However, of course, the investigation of the quark mass matrix is still not completed as well as that of the neutrino mass matrix. The mass matrix studies will be continued still hereafter.

The purpose of the present note is not to discuss the physics on such the mass matrix models, but to give a list of collected mass matrix formulae for convenience of the model-builders.

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# Notations and Conventions

We define Dirac mass matrix  $M_f$  for fermions  $f = (f_1, f_2, \dots)$  by

$$H_{mass} = \bar{f}_L M_f f_R + \bar{f}_R M_f^\dagger f_L. \quad (0.1)$$

The mass matrices  $M_f$  are diagonalized as

$$U_{fL}^\dagger M_f U_{fR} = D_f \equiv \text{diag}(m_{f1}, m_{f2}, \dots). \quad (0.2)$$

The mixing matrices  $U_{fL}$  and  $U_{fR}$  are given by

$$U_{fL}^\dagger M_f M_f^\dagger U_{fL} = D_f^2, \quad (0.3)$$

$$U_{fR}^\dagger M_f^\dagger M_f U_{fR} = D_f^2, \quad (0.4)$$

respectively.

For example, quark mass matrices  $(M_u, M_d)$  are diagonalized as

$$U_{uL}^\dagger M_u U_{uR} = D_u \equiv \text{diag}(m_u, m_c, m_t), \quad (0.5)$$

$$U_{dL}^\dagger M_d U_{dR} = D_d \equiv \text{diag}(m_d, m_s, m_b), \quad (0.6)$$

Then, the Kobayashi-Maskawa (CKM) matrix  $V$  is given by

$$V = U_{uL}^\dagger U_{dL}, \quad (0.7)$$

for the case  $f_L = q_L \equiv (u, d)_L$ .

For a neutrino Majorana mass matrix  $M_\nu$  ( $M_\nu^T = M_\nu$ ), which is defined by

$$H_{mass} = \bar{\nu}_L M_\nu \nu_L^* \quad (0.8)$$

the mass matrix  $M_\nu$  is diagonalized as

$$U_{\nu L}^\dagger M_\nu U_{\nu L}^* = D_\nu \equiv \text{diag}(m_{\nu 1}, m_{\nu 2}, m_{\nu 3}) \quad (0.9)$$

The Maki-Nakagawa-Sakata matrix  $U$  is given by

$$U = U_{eL}^\dagger U_{\nu L}, \quad (0.10)$$

where  $U_{eL}$  is defined by

$$U_{eL}^\dagger M_e U_{eR} = D_e \equiv \text{diag}(m_e, m_\mu, m_\tau) \quad (0.11)$$

# 1 General Formulae

General formulae for arbitrary  $n \times n$  matrices are given. For some of them, it is assumed that they are Hermitian.

## 1.1 Trace

### Definition

Trace of an  $n \times n$  matrix  $A$  is defined as

$$\text{Tr}A = \sum_i^n A_{ii}. \quad (1.1.1)$$

### Related formulae

Matrices  $A, B, \dots$  are satisfied the following equations:

$$\text{Tr}(A + B) = \text{Tr}A + \text{Tr}B, \quad (1.1.2)$$

$$\text{Tr}(aA) = a\text{Tr}A \quad (a : c - \text{number}), \quad (1.1.3)$$

$$\text{Tr}(AB) = \text{Tr}(BA), \quad (1.1.4)$$

$$\text{Tr}(PBP^{-1}) = \text{Tr}A. \quad (1.1.4)$$

## 1.2 Transpose

### Definition

Transpose  $A^T$  of an  $n \times n$  matrix  $A$  is defined by

$$(A^T)_{ij} = A_{ji}. \quad (1.2.1)$$

### Related formulae

$$(A + B)^T = A^T + B^T, \quad (1.2.2)$$

$$(aA)^T = aA^T \quad (a : c - \text{number}), \quad (1.2.3)$$

$$(AB)^T = B^T A^T, \quad (1.2.4)$$

$$\text{Tr}(A^T) = \text{Tr}A, \quad (1.2.5)$$

$$(A^T)^T = A. \quad (1.2.6)$$

## 1.3 Hermitian conjugate

### Definition

Hermite conjugate  $A^\dagger$  an  $n \times n$  matrix  $A$  is defined by

$$A^\dagger = (A^*)^T. \quad (1.3.1)$$

## Related formulae

$$(A + B)^\dagger = A^\dagger + B^\dagger, \quad (1.3.2)$$

$$(aA)^\dagger = a^* A^\dagger \quad (a : c\text{-number}), \quad (1.3.3)$$

$$(AB)^\dagger = B^\dagger A^\dagger, \quad (1.3.4)$$

$$\text{Tr}(A^\dagger) = (\text{Tr}A)^*, \quad (1.3.5)$$

$$(A^\dagger)^\dagger = A. \quad (1.3.6)$$

The following formula<sup>1</sup> is also useful:

$$\text{Tr} \ln(A + B) = \text{Tr} \ln A + \int_0^1 dz \text{Tr} \frac{1}{A + zB} B. \quad (1.3.7)$$

## 1.4 Exponential

### Definition

Exponential  $e^A$  is defined by

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n = 1 + A + \frac{1}{2!} A^2 + \frac{1}{3!} A^3 + \dots. \quad (1.4.1)$$

### Related formulae

$$(e^A)^{-1} = e^{-A}, \quad (1.4.2)$$

$$P e^A P^{-1} = e^{P A P^{-1}}, \quad (1.4.3)$$

$$e^{A^T} = (e^A)^T, \quad e^{A^\dagger} = (e^A)^\dagger. \quad (1.4.4)$$

For  $AB - BA = 0$ , we obtain

$$e^{A+B} = e^A e^B, \quad (1.4.5)$$

but, for  $AB - BA \neq 0$ , we obtain

$$e^A e^B = \exp \left( A + B + \frac{1}{2!} [A, B] + \frac{1}{3!} \frac{1}{2} [A - B, [A, B]] - \frac{1}{4!} [B, [A, [A, B]]] + \dots \right), \quad (1.4.6)$$

(the Baker-Campbell-Hausdorff formula). Also, we can obtain

$$e^{aB} A e^{-aB} = A + \frac{a}{1!} [B, A] + \frac{a^2}{2!} [B, [B, A]] + \dots. \quad (1.4.7)$$

## 1.5 Determinant

### Definition

Determinant  $\det A$  is defined by

$$\det A = |a_{ij}| = \sum \eta(R) a_{1r_1} a_{2r_2} \dots a_{nr_n}, \quad (1.5.1)$$

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<sup>1</sup>O. Cheyette, Phys. Rev. Lett. **55**, 2394 (1985).

where  $R = (r_1, r_2, \dots, r_n)$  and  $\eta(R) = +1$  or  $\eta(R) = -1$  according as  $R$  is even or odd permutation of  $(1, 2, \dots, n)$ , respectively.

### Related formulae

$$\det A^T = \det A, \quad (1.5.2)$$

$$\det(AB) = \det(BA) = \det A \det B = \det B \det A, \quad (1.5.3)$$

$$\det A^{-1} = (\det A)^{-1}, \quad (1.5.4)$$

$$\det e^A = e^{\text{Tr}A}, \quad (1.5.5)$$

$$\log \det A = \text{Tr} \log A. \quad (1.5.6)$$

For a  $2n \times 2n$  matrix  $M$ , the following formula is also useful: The determinant of the matrix

$$M = \begin{pmatrix} A & C \\ D & B \end{pmatrix} \quad (1.5.7)$$

is given by

$$\det M = \det(A - CB^{-1}D) \cdot \det B = \det A \cdot \det(B - DA^{-1}C), \quad (1.5.8)$$

because

$$\begin{pmatrix} A & C \\ D & B \end{pmatrix} = \begin{pmatrix} A - CB^{-1}D & CB^{-1} \\ 0 & B \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ D & B \end{pmatrix} = \begin{pmatrix} A & 0 \\ D & B \end{pmatrix} \begin{pmatrix} \mathbf{1} & A^{-1}C \\ 0 & B - DA^{-1}C \end{pmatrix}. \quad (1.5.9)$$

### 1.6 Inverse

The inverse of the matrix (1.5.6) is given by

$$\begin{pmatrix} A & C \\ D & B \end{pmatrix}^{-1} = \begin{pmatrix} (A - CB^{-1}D)^{-1} & (D - BC^{-1}A)^{-1} \\ (C - AD^{-1}B)^{-1} & (B - DA^{-1}C)^{-1} \end{pmatrix}. \quad (1.6.1)$$

This is easily confirmed by calculating  $MM^{-1}$ .

### 1.7 Series expansion

When  $O(A) \gg O(B)$ , the matrix  $(A + B)^{-1}$  is expanded as follows:

$$\begin{aligned} \frac{1}{A+B} &= \frac{1}{A} - \frac{1}{A}B\frac{1}{A+B} \\ &= \frac{1}{A} - \frac{1}{A}B\frac{1}{A} + \frac{1}{A}B\frac{1}{A}B\frac{1}{A} - \dots \end{aligned} \quad (1.7.1)$$

### 1.8 Expansion of a mass matrix in terms of rank-1 matrices

[Theorem]

A Hermitian mass matrix  $M$  is expressed as

$$M = \sum_{i=1}^n m_i A_i, \quad (1.8.1)$$

where  $A_i$  are rank-1 matrices, and those satisfy the relations

$$A_i A_j = \delta_{ij} A_i. \quad (1.8.2)$$

**[Proof]**

The Hermitian mass matrix  $M$  is diagonalized as

$$U^\dagger M U = D \equiv \text{diag}(m_1, m_2, m_3, \dots, m_n) = \sum_{i=1}^n m_i D_i, \quad (1.8.3)$$

where

$$\begin{aligned} D_1 &= \text{diag}(1, 0, 0), \\ D_2 &= \text{diag}(0, 1, 0), \\ D_3 &= \text{diag}(0, 0, 1), \end{aligned} \quad (1.4.4)$$

and so on, so that

$$D_i D_j = \delta_{ij} D_i. \quad (1.8.5)$$

Therefore,  $M$  is expressed as

$$M = U D U^\dagger = \sum_{i=1}^n m_i A_i, \quad (1.8.6)$$

where

$$A_i = U D_i U^\dagger, \quad (1.8.7)$$

so that the rank-1 matrices  $A_i$  satisfy the relations

$$A_i A_j = U D_i U^\dagger U D_j U^\dagger = U \delta_{ij} D_i U^\dagger = \delta_{ij} A_i. \quad (1.8.8)$$

**[Corollary]**

A Hermitian matrix  $H_L = M M^\dagger$  (and also  $H_R = M^\dagger M$ ) is expressed as

$$H_{L(R)} = \sum_{i=1}^n m_i^2 A_i, \quad (1.8.9)$$

where rank-1 matrices  $A_i$  satisfy the relations

$$A_i A_j = \delta_{ij} A_i. \quad (1.8.10)$$

**[Proof]**

$$H_{L(R)} = U_{L(R)} D^2 U_{L(R)}^\dagger = \sum_{i=1}^n m_i^2 A_i \quad (1.8.11)$$

where

$$A_i = U_{L(R)} D_i U_{L(R)}^\dagger \quad (1.8.12)$$

so that from the discussion similar to (1.8.8) we can confirm the relations (1.8.10).

**[Corollary]**

The relation (1.8.1) with (1.8.2) and (1.8.9) with (1.8.10) are applicable to the neutrino mass matrix  $M_\nu$ , which is diagonalized as

$$U_\nu^\dagger M_\nu U_\nu^* = D_\nu. \quad (1.8.13)$$

**[Proof]**

If  $M_\nu$  is a real matrix, the matrices  $A_i$  in the expression

$$M_\nu = \sum_{i=1}^n m_i A_i, \quad (1.8.14)$$

are given by

$$A_i = U_\nu D_i U_\nu^T, \quad (1.8.15)$$

where  $U_\nu$  is orthogonal matrix.

If  $M_\nu$  includes complex parameters, the matrices  $A_i$  in the expression

$$M_\nu M_\nu^\dagger = \sum_{i=1}^n m_i^2 A_i, \quad (1.8.16)$$

are given by

$$A_i = U_\nu D_i U_\nu^\dagger, \quad (1.8.17)$$

for  $H_L = M_\nu M_\nu^\dagger$ .

## 2 Fierz Transformation

The Fierz transformation is very useful for calculating a trace of a matrix. The Fierz transformation for  $n \times n$  Hermitian matrices is reviewed.

### 2.1 General formula

An arbitrary  $n \times n$  Hermitian matrix  $A$  has  $n^2$  independent parameters, so that it can be expanded by  $n^2$  independent matrices  $F_i$  ( $i = 0, 1, 2, \dots, n^2 - 1$ ) as follows:

$$A = \sum_{i=0}^{n^2-1} a_i F_i, \quad (2.1.1)$$

where  $F_i$  is normalized as

$$\text{Tr}(F_i F_j) = k \delta_{ij}. \quad (2.1.2)$$

Then, by solving (2.1.1) inversely, we can obtain

$$a_i = \frac{1}{k} \text{Tr}(A F_i). \quad (2.1.3)$$

By substituting (2.1.3) for (2.1.1), we obtain

$$A = \frac{1}{k} \sum_{i=0}^{n^2-1} \text{Tr}(A F_i) F_i. \quad (2.1.4)$$

When we consider a case

$$A_{\alpha\beta} = \delta_{\alpha\rho} \delta_{\sigma\beta}, \quad (2.1.5)$$

the relation (2.1.4) leads to a formula

$$\delta_{\alpha\rho} \delta_{\sigma\beta} = \frac{1}{k} \sum_{i=0}^{n^2-1} (F_i)_{\alpha\beta} (F_i)_{\sigma\rho}. \quad (2.1.6)$$

For arbitrary  $n \times n$  Hermitian matrices  $A$  and  $B$ , the following formulae are derived by using Eq.(2.1.6):

$$\text{Tr}(A) \delta_{\alpha\beta} = \frac{1}{k} \sum_{i=0}^{n^2-1} (F_i A F_i)_{\alpha\beta}, \quad (2.1.7)$$

$$\text{Tr}(A) B = \frac{1}{k} \sum_{i=0}^{n^2-1} F_i A F_i B = \frac{1}{k} \sum_{i=0}^{n^2-1} B F_i A F_i, \quad (2.1.8)$$

$$\text{Tr} A \text{Tr} B = \frac{1}{k} \sum_{i=0}^{n^2-1} \text{Tr}(F_i A F_i B), \quad (2.1.9)$$

$$\text{Tr}(AB) = \frac{1}{k} \sum_{i=0}^{n^2-1} \text{Tr}(A F_i) \text{Tr}(F_i B). \quad (2.1.10)$$

Usually, the matrix  $F_0$  is chosen as

$$F_0 = \sqrt{\frac{k}{n}} \mathbf{1}, \quad (2.1.11)$$

so that the trace of  $F_k$  is zero from the definition of  $F_k$ , (2.1.2),

$$\text{Tr}(F_k) = 0 \quad (k \neq 0). \quad (2.1.12)$$

By choosing  $A = F_k$  ( $k \neq 0$ ) in Eq.(2.1.7), we obtain

$$\sum_{i=0}^{n^2-1} F_i F_k F_i = -\frac{k}{n} F_k \quad (k \neq 0). \quad (2.1.13)$$

Also, the formula (2.1.10) is written as

$$\text{Tr}(AB) = \frac{1}{n} \text{Tr}(A) \text{Tr}(B) + \frac{1}{k} \sum_{i=1}^{n^2-1} \text{Tr}(A F_i) \text{Tr}(F_i B). \quad (2.1.14)$$

The formula (2.1.14) is known as the Fierz transformation.

If the matrices  $F_i$  are matrices of the ad joint representation in  $U(n)$ , they satisfy the relation

$$\text{Tr}(F_i F_j) = T(R) \delta_{ij}, \quad (2.1.15)$$

where

$$k = T(R) = \frac{1}{2}, \quad (2.1.16)$$

and the Casimir invariant  $C_2(R)$  is given by

$$C_2(R) = \sum_{i=0}^{n^2-1} F_i^2 \quad (2.1.17)$$

where

$$C_2(R) = \frac{n}{2} \quad \text{for } U(n), \quad (2.1.18)$$

$$C_2(R) = \frac{n^2-1}{2n} n \quad \text{for } SU(n). \quad (2.1.19)$$

## 2.2 Case of $2 \times 2$ matrices

We can choose a unit matrix  $\sigma_0$  and three Pauli matrices  $\sigma_k$  as 4 matrices  $F_i$ :

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (2.2.1)$$

which satisfy

$$\text{Tr}(\sigma_i \sigma_j) = 2 \delta_{ij}, \quad (2.2.2)$$

so that we regard  $F_i$  as  $F_i = (1/2)\sigma_i$  in order to take  $k = 1/2$ . Then, we obtain the formula

$$\text{Tr}(AB) = \frac{1}{2} \text{Tr}(A) \text{Tr}(B) + \frac{1}{2} \sum_{i=1}^3 \text{Tr}(A \sigma_i) \text{Tr}(\sigma_i B). \quad (2.2.3)$$

### 2.3 Case of $3 \times 3$ matrices

For the case of  $3 \times 3$  matrices, we choose a unit matrix  $\mathbf{1}$  and Gell-Mann's  $\lambda$  matrices as 9 matrices  $F_i$ :

$$\begin{aligned} \lambda_0 &= \sqrt{\frac{2}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & \lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \\ \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, & \lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, & \lambda_8 &= \sqrt{\frac{1}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \end{aligned} \quad (2.3.1)$$

which satisfy

$$\text{Tr}(\lambda_i \lambda_j) = 2\delta_{ij}, \quad (2.3.2)$$

so that we regard  $F_i$  as  $F_i = (1/2)\lambda_i$  in order to take  $k = 1/2$ . Then, we obtain the formula

$$\text{Tr}(AB) = \frac{1}{3}\text{Tr}(A)\text{Tr}(B) + \frac{1}{2}\sum_{i=k}^8 \text{Tr}(A\lambda_k)\text{Tr}(\lambda_k B), \quad (2.2.3)$$

from Eq.(2.2.3). For fermion fields  $\psi_a$  ( $a = 1, 2, 3, 4$ ), we obtain

$$(\bar{\psi}_1 \psi_2)(\bar{\psi}_3 \psi_4) = -\frac{1}{2}\sum_{k=1}^8 (\bar{\psi}_1 \lambda_k \psi_4)(\bar{\psi}_3 \lambda_k \psi_2), \quad (2.2.4)$$

where the sign  $-1$  comes from the interchange of the fermion fields. The following formula is also useful:

$$\frac{4}{3}(\bar{\psi}_1 \psi_2)(\bar{\psi}_3 \psi_4) - \frac{1}{2}\left(\sum_{k=1}^8 (\bar{\psi}_1 \lambda_k \psi_2)(\bar{\psi}_3 \lambda_k \psi_4)\right) = \frac{4}{3}(\bar{\psi}_1 \psi_4)(\bar{\psi}_3 \psi_2) - \frac{1}{2}\left(\sum_{k=1}^8 (\bar{\psi}_1 \lambda_k \psi_4)(\bar{\psi}_3 \lambda_k \psi_2)\right). \quad (2.2.5)$$

Note that, in the expressions (2.2.4) and (2.2.5), factors from  $\gamma$  matrices for four spinor fields  $\psi_a$  must also be taken into consideration as stated in the next section.

### 2.4 Case of $4 \times 4$ matrices

Any  $4 \times 4$  matrix can be expanded by 16 independent matrix  $\Gamma_i$  ( $i = 0, 1, \dots, 15$ ):

$$\Gamma_i = (\mathbf{1}, \gamma_5, \gamma^0, i\gamma^k, i\gamma^0\gamma_5, \gamma^k\gamma_5, \sigma^{k\ell}, \sigma^{0k}), \quad (2.4.1)$$

where  $k, \ell = 1, 2, 3$ , and  $\gamma^\mu$  are the so-called  $\gamma$  matrices

$$\gamma^0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \quad (2.4.2)$$

and

$$\gamma_5 = \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}. \quad (2.4.3)$$

The matrix  $\sigma^{\mu\nu}$  is defined by

$$\sigma^{\mu\nu} = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu). \quad (2.4.4)$$

Then, the matrices  $\Gamma_i$  satisfy

$$\Gamma_i^2 = \mathbf{1}, \quad (2.4.5)$$

$$\text{Tr}(\Gamma_i\Gamma_j) = 4\delta_{ij}. \quad (2.4.6)$$

Therefore, the formula (2.1.6) is explicitly given by

$$\delta_{\alpha\rho}\delta_{\sigma\beta} = \frac{1}{4} \sum_{i=0}^{15} (\Gamma_i)_{\alpha\beta} (\Gamma_i)_{\sigma\rho}. \quad (2.4.7)$$

Here, the sum is symbolically expressed as

$$\sum_i \Gamma_i \otimes \Gamma_i = \mathbf{1} \otimes \mathbf{1} + \gamma_5 \otimes \gamma_5 + \gamma_\mu \otimes \gamma^\mu + \gamma_\mu\gamma_5 \otimes \gamma^\mu\gamma_5 + \sigma_{\mu\nu} \otimes \sigma^{\mu\nu}. \quad (2.4.8)$$

For four spinor fermion fields  $\psi_a$ , Eq.(2.4.7) gives

$$(\bar{\psi}_1\psi_2)(\bar{\psi}_3\psi_4) = -\frac{1}{4} \sum_{i=1}^{15} (\bar{\psi}_1\Gamma_i\psi_4)(\bar{\psi}_3\Gamma_i\psi_2) \quad (2.5.9)$$

where the factor  $-1$  from a fermion field exchange has already be taken into consideration. A more general case is given as follows:

$$(\bar{\psi}_1\Gamma_A\psi_2)(\bar{\psi}_3\Gamma_B\psi_4) = -\frac{1}{4} \sum_{i=1}^{15} (\bar{\psi}_1\Gamma_A\Gamma_i\Gamma_B\psi_4)(\bar{\psi}_3\Gamma_i\psi_2). \quad (2.5.10)$$

The typical cases are as follows:

$$\begin{aligned} (\bar{\psi}_1\gamma_\mu\psi_2)(\bar{\psi}_3\gamma^\mu\psi_4) &= -(\bar{\psi}_1\psi_4)(\bar{\psi}_3\psi_2) + (\bar{\psi}_1\gamma_5\psi_4)(\bar{\psi}_3\gamma_5\psi_2) \\ &+ \frac{1}{2}(\bar{\psi}_1\gamma_\mu\psi_4)(\bar{\psi}_3\gamma^\mu\psi_2) + \frac{1}{2}(\bar{\psi}_1\gamma_\mu\gamma_5\psi_4)(\bar{\psi}_3\gamma^\mu\gamma_5\psi_2) \end{aligned} \quad (2.5.11)$$

$$(\bar{\psi}_1\gamma_\mu(1 \pm \gamma_5)\psi_2)(\bar{\psi}_3\gamma^\mu(1 \pm \gamma_5)\psi_4) = (\bar{\psi}_1\gamma_\mu(1 \pm \gamma_5)\psi_4)(\bar{\psi}_3\gamma^\mu(1 \pm \gamma_5)\psi_2), \quad (2.5.12)$$

$$(\bar{\psi}_1\gamma_\mu(1 \pm \gamma_5)\psi_2)(\bar{\psi}_3\gamma^\mu(1 \mp \gamma_5)\psi_4) = -2(\bar{\psi}_1\gamma_\mu(1 \mp \gamma_5)\psi_4)(\bar{\psi}_3\gamma^\mu(1 \pm \gamma_5)\psi_2), \quad (2.5.13)$$

$$\begin{aligned} (\bar{\psi}_1(1 \pm \gamma_5)\psi_2)(\bar{\psi}_3(1 \pm \gamma_5)\psi_4) &= -\frac{1}{2}(\bar{\psi}_1(1 \pm \gamma_5)\psi_4)(\bar{\psi}_3(1 \pm \gamma_5)\psi_2) \\ &- \frac{1}{8}(\bar{\psi}_1(1 \pm \gamma_5)\sigma_{\mu\nu}\psi_4)(\bar{\psi}_3(1 \pm \gamma_5)\sigma^{\mu\nu}\psi_2), \end{aligned} \quad (2.5.14)$$

$$\begin{aligned} (\bar{\psi}_1\gamma_\mu(1 \pm \gamma_5)\psi_2)(\bar{\psi}_3\gamma^\mu\psi_4) &= -\frac{1}{2}(\bar{\psi}_1(1 \mp \gamma_5)\psi_4)(\bar{\psi}_3(1 \pm \gamma_5)\psi_2) \\ &+ \frac{1}{2}(\bar{\psi}_1\gamma_\mu(1 \pm \gamma_5)\psi_4)(\bar{\psi}_3\gamma^\mu(1 \pm \gamma_5)\psi_2), \end{aligned} \quad (2.5.15)$$

where we have used the formulae for the  $\gamma$  matrices:

$$\gamma_\mu\gamma^\mu, \quad (2.5.16)$$

$$\gamma_\mu\gamma_\lambda\gamma^\mu = -2\gamma_\lambda, \quad (2.5.17)$$

$$\gamma_\mu \gamma_\lambda \gamma_\rho \gamma^\mu = 4g_{\lambda\rho}, \quad (2.5.18)$$

$$\gamma_\mu \gamma_\lambda \gamma_\rho \gamma_\sigma \gamma^\mu = -2\gamma_\sigma \gamma_\rho \gamma_\lambda, \quad (2.5.19)$$

$$\sigma_{\mu\nu} \gamma_5 = \frac{i}{2} \varepsilon_{\mu\nu\rho\sigma} \sigma^{\rho\sigma}, \quad (2.5.20)$$

$$i\sigma_{\mu\nu} \gamma_\rho = -i\varepsilon_{\mu\nu\rho\lambda} \gamma^\lambda \gamma_5 + \gamma_\mu g_{\nu\rho} - \gamma_\nu g_{\mu\rho}, \quad (2.5.21)$$

( $\varepsilon_{0123} = +1$ ).

### 3 Diagonalization of an $(m + n) \times (m + n)$ Matrix

We obtain a seesaw type expression from an  $(m + n) \times (m + n)$  matrix.

#### 3.1 Formulation

The  $(m + n) \times (m + n)$  matrix

$$M = \begin{pmatrix} A & C \\ D & B \end{pmatrix} \quad (3.1.1)$$

is diagonalized by the following mixing matrices  $U_L$  and  $U_R$ :

$$U_L^\dagger M U_R = \begin{pmatrix} D_m & 0 \\ 0 & D_n \end{pmatrix} \quad (3.1.2)$$

$$U_L = \begin{pmatrix} (\mathbf{1} + \rho_L \rho_L^\dagger)^{-\frac{1}{2}} & -\rho_L (\mathbf{1} + \rho_L^\dagger \rho_L)^{-\frac{1}{2}} \\ \rho_L^\dagger (\mathbf{1} + \rho_L \rho_L^\dagger) & (\mathbf{1} + \rho_L^\dagger \rho_L)^{-\frac{1}{2}} \end{pmatrix}, \quad (3.1.3)$$

$$U_R = \begin{pmatrix} (\mathbf{1} + \rho_R \rho_R^\dagger)^{-\frac{1}{2}} & -\rho_R (\mathbf{1} + \rho_R^\dagger \rho_R)^{-\frac{1}{2}} \\ \rho_R^\dagger (\mathbf{1} + \rho_R \rho_R^\dagger) & (\mathbf{1} + \rho_R^\dagger \rho_R)^{-\frac{1}{2}} \end{pmatrix}, \quad (3.1.4)$$

where

- $A$  :  $m \times m$  matrix,
- $B$  :  $n \times n$  matrix,
- $C$  :  $m \times n$  matrix,
- $D$  :  $n \times m$  matrix,
- $D_m$  :  $m \times m$  matrix,
- $D_n$  :  $n \times n$  matrix,
- $\rho_{L/R}$  :  $m \times n$  matrix.

Here, the matrices  $D_m$  and  $D_n$  do not mean that they are diagonal.

The expressions (3.1.3) and (3.1.4) automatically satisfy the unitary conditions

$$U_{L/R} U_{L/R}^\dagger = U_{L/R}^\dagger U_{L/R} = \mathbf{1}. \quad (3.1.5)$$

Here, for derivation of  $U U^\dagger = \mathbf{1}$ , we have used the following relations

$$\rho (\mathbf{1} + \rho^\dagger \rho)^{-1} = (\mathbf{1} + \rho \rho^\dagger)^{-1} \rho. \quad (3.1.6)$$

$$\rho^\dagger (\mathbf{1} + \rho \rho^\dagger)^{-1} = (\mathbf{1} + \rho^\dagger \rho)^{-1} \rho^\dagger. \quad (3.1.7)$$

#### 3.2 A case of an $(m + n) \times (m + n)$ Hermitian mass matrix

When an  $(m + n) \times (m + n)$  mass matrix  $M$  is Hermitian, the mass matrix is expressed as

$$M = \begin{pmatrix} M_1 & m \\ m^\dagger & M_2 \end{pmatrix}, \quad (3.2.1)$$

and the matrix (3.2.1) is diagonalized as follows:<sup>2</sup>

$$U^\dagger M U = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \quad (3.2.2)$$

where  $U$  is defined by

$$U = \begin{pmatrix} (\mathbf{1} + \rho\rho^\dagger)^{-\frac{1}{2}} & -\rho(\mathbf{1} + \rho^\dagger\rho)^{-\frac{1}{2}} \\ \rho^\dagger(\mathbf{1} + \rho\rho^\dagger)^{-\frac{1}{2}} & (\mathbf{1} + \rho^\dagger\rho)^{-\frac{1}{2}} \end{pmatrix}. \quad (3.2.3)$$

Then, we obtain

$$(U^\dagger M U)_{12} = (\mathbf{1} + \rho\rho^\dagger)^{-\frac{1}{2}}(\rho M_2 - M_1\rho + m - \rho m^\dagger\rho)(\mathbf{1} + \rho^\dagger\rho)^{-\frac{1}{2}}, \quad (3.2.4)$$

$$(U^\dagger M U)_{11} = (\mathbf{1} + \rho\rho^\dagger)^{-\frac{1}{2}}(M_1 + \rho M_2\rho^\dagger + m\rho^\dagger + \rho m^\dagger)(\mathbf{1} + \rho\rho^\dagger)^{-\frac{1}{2}}, \quad (3.2.5)$$

$$(U^\dagger M U)_{22} = (\mathbf{1} + \rho^\dagger\rho)^{-\frac{1}{2}}(\rho^\dagger M_1\rho + M_2 - \rho^\dagger m - m^\dagger\rho)(\mathbf{1} + \rho^\dagger\rho)^{-\frac{1}{2}}, \quad (3.2.6)$$

From Eq.(3.2.4), we get the condition for the matrix  $\rho$ ,

$$\rho M_2 - M_1\rho + m - \rho m^\dagger\rho = 0. \quad (3.2.7)$$

Then, we obtain expressions for  $D_1$  and  $D_2$ ,

$$D_1 = (\mathbf{1} + \rho\rho^\dagger)^{-\frac{1}{2}}(M_1 + \rho m^\dagger)(\mathbf{1} + \rho\rho^\dagger)^{\frac{1}{2}}, \quad (3.2.8)$$

$$D_2 = (\mathbf{1} + \rho^\dagger\rho)^{\frac{1}{2}}(M_2 - m^\dagger\rho)(\mathbf{1} + \rho^\dagger\rho)^{-\frac{1}{2}}. \quad (3.2.9)$$

### 3.3 Seesaw approximation

We consider a case

$$O(M_2) \gg O(m) \gg O(M_1). \quad (3.3.1)$$

From Eq.(3.2.7), we approximate

$$\rho = (m - M_1\rho)(M_2 - m^\dagger\rho)^{-1} \simeq (m + M_1 m M_2^{-1})M_2^{-1}(\mathbf{1} - m^\dagger m M_2^{-2}). \quad (3.3.2)$$

Since

$$\rho\rho^\dagger \simeq m M_2^{-2} m^\dagger, \quad (3.3.3)$$

$$(\mathbf{1} + \rho\rho^\dagger)^{-\frac{1}{2}} \simeq \mathbf{1} - \frac{1}{2} m M_2^{-2} m^\dagger, \quad (3.3.4)$$

we obtain

$$\begin{aligned} D_1 &\simeq \left(\mathbf{1} - \frac{1}{2} m M_2^{-2} m^\dagger\right) (M_1 - m M_2^{-1} m^\dagger) \left(\mathbf{1} - \frac{1}{2} m M_2^{-2} m^\dagger\right) \\ &\simeq M_1 - \frac{1}{2} \left(m M_2^{-2} m^\dagger M_1 + M_1 m M_2^{-2} m^\dagger\right) \\ &\quad - m M_2^{-1} m^\dagger + \frac{1}{2} \left(m M_2^{-1} m^\dagger m M_2^{-2} m^\dagger + m M_2^{-2} m^\dagger m M_2^{-1} m^\dagger\right), \end{aligned} \quad (3.3.5)$$

from Eq. (3.2.8). And also, from Eq. (3.2.9) and

$$\rho^\dagger\rho \simeq M_2^{-1} m^\dagger m M_2^{-1}, \quad (3.3.6)$$

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<sup>2</sup>See Appendix in Y. Koide and H. Fusaoka, Z. Phys. **C71**, 459 (1996).

we obtain

$$\begin{aligned}
D_2 &\simeq \left( \mathbf{1} + \frac{1}{2} M_2^{-1} m^\dagger m M_2^{-1} \right) (M_2 + m^\dagger m M_2^{-1}) \left( \mathbf{1} - \frac{1}{2} M_2^{-1} m^\dagger m M_2^{-1} \right) \\
&\simeq M_2 + \frac{1}{2} m^\dagger m M_2^{-1} + \frac{1}{2} M_2^{-1} m^\dagger m.
\end{aligned} \tag{3.3.7}$$

The mixing matrix  $U$ , (3.2.3), is given by

$$U \simeq \begin{pmatrix} \mathbf{1} - \frac{1}{2} m M_2^{-2} m^\dagger & m M_2^{-1} \left( \mathbf{1} - \frac{1}{2} M_2^{-1} m^\dagger m M_2^{-1} \right) \\ -M_2^{-1} m^\dagger \left( \mathbf{1} - \frac{1}{2} m M_2^{-2} m^\dagger \right) & \mathbf{1} - \frac{1}{2} M_2^{-1} m^\dagger m M_2^{-1} \end{pmatrix} \tag{3.3.8}$$

## 4 General Formulae for $3 \times 3$ Matrices

General formulae for arbitrary  $3 \times 3$  matrices are given.

### 4.1 Determinant

For a  $3 \times 3$  matrix  $A$ , the determinant of  $A$  is given by

$$\begin{aligned} \det A &= \frac{1}{3} \text{Tr}(A^3) - \frac{1}{2} \text{Tr}(A^2) \text{Tr} A + \frac{1}{6} (\text{Tr} A)^3 \\ &= \frac{1}{3} [\text{Tr}(A^3) - (\text{Tr} A)^3] - \frac{1}{2} [\text{Tr}(A^2) - (\text{Tr} A)^2] \text{Tr} A. \end{aligned} \quad (4.1.1)$$

For a traceless matrix  $A_0$

$$A_0 = A - \frac{1}{3} \text{Tr}(A) \cdot \mathbf{1}, \quad (4.1.2)$$

we obtain

$$\det A_0 = \frac{1}{3} \text{Tr}(A_0^3). \quad (4.1.3)$$

From Eq. (4.1.2), we can express  $\det A$  as

$$\det A - \det A_0 = \frac{1}{27} (\text{Tr} A)^3 - \frac{1}{6} \text{Tr}(A_0^2) \text{Tr} A = \frac{5}{54} (\text{Tr} A)^3 - \frac{1}{6} \text{Tr}(A^2) \text{Tr} A. \quad (4.1.4)$$

### 4.2 Trace

For arbitrary  $3 \times 3$  matrices  $A$  and  $B$ , the trace of the matrix  $AB$  is given by

$$\text{Tr}(AB) = \frac{1}{3} \text{Tr} A \text{Tr} B + \frac{1}{2} \sum_{a=1}^8 \text{Tr}(A \lambda_a) \text{Tr}(B \lambda_a), \quad (4.2.1)$$

where  $\lambda_a$  are Gell-Mann's  $\lambda$  matrices,

$$\begin{aligned} \lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & \lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \\ \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, & \lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \\ \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \end{aligned} \quad (4.2.2)$$

which satisfy the relations

$$\text{Tr}(\lambda_a) = 0, \quad (4.2.3)$$

$$\text{Tr}(\lambda_a \lambda_b) = 2\delta_{ab}. \quad (4.2.4)$$

If the  $3 \times 3$  matrices  $A$ ,  $B$ ,  $C$  and  $D$  have the same traces, i.e.

$$\text{Tr}A = \text{Tr}B = \text{Tr}C = \text{Tr}D, \quad (4.2.5)$$

those satisfy the relation

$$\begin{aligned} & \text{Tr}(ABCD + ACBD + ABDC + ACDB + ADBC + ADCB) \\ &= \text{Tr}(AB)\text{Tr}(CD) + \text{Tr}(AC)\text{Tr}(BD) + \text{Tr}(AD)\text{Tr}(BC). \end{aligned} \quad (4.2.6)$$

For the case with  $A = C$  and  $B = D$ , we obtain

$$4\text{Tr}(A^2B^2) + 2\text{Tr}(ABAB) = 2\text{Tr}(AB)\text{Tr}(AB) + \text{Tr}(A^2)\text{Tr}(B^2). \quad (4.2.7)$$

For the case of  $A = B$ , we obtain

$$\text{Tr}(A^4) = \frac{1}{2} \left( \text{Tr}(A^2) \right)^2. \quad (4.2.8)$$

If the matrix  $A$  is not traceless, then we can define a traceless matrix  $A_0$  as

$$A_0 = A - \frac{1}{3}\text{Tr}(A) \cdot \mathbf{1}, \quad (4.2.9)$$

so that we can derive formulae for non-traceless matrices correspondingly to Eqs. (4.2.6) – (4.2.8). Here, it is useful to use a formula

$$A_0B_0 = AB - \frac{1}{3}\text{Tr}(A) \cdot B - \frac{1}{3}A \cdot \text{Tr}B + \frac{1}{9}\text{Tr}(A)\text{Tr}(B) \cdot \mathbf{1}. \quad (4.2.10)$$

For example, we obtain

$$\text{Tr}[A_0B_0] = \text{Tr}[AB] - \frac{1}{3}\text{Tr}[A]\text{Tr}[B], \quad (4.2.11)$$

$$\begin{aligned} \text{Tr}[A_0B_0C_0] &= \text{Tr}[ABC] - \frac{1}{3}\text{Tr}[A]\text{Tr}[BC] - \frac{1}{3}\text{Tr}[B]\text{Tr}[CA] \\ &\quad - \frac{1}{3}\text{Tr}[C]\text{Tr}[AB] + \frac{2}{9}\text{Tr}[A]\text{Tr}[B]\text{Tr}[C], \end{aligned} \quad (4.2.12)$$

$$\text{Tr}[A_0^3] = \text{Tr}[A^3] - \text{Tr}[A]\text{Tr}[A^2] + \frac{2}{9}(\text{Tr}[A])^3, \quad (4.2.13)$$

and so on.

### 4.3 Hermitian matrix

**[Theorem]**

An arbitrary Hermitian matrix  $A$  satisfies the relation

$$A^3 - \text{Tr}A \cdot A^2 + \frac{1}{2} \left[ (\text{Tr}A)^2 - \text{Tr}(A^2) \right] A - \det A = 0, \quad (4.3.1)$$

**[Proof]**

The Hermitian matrix  $A$  is diagonalized as

$$U^\dagger AU = D \equiv \text{diag}(a_1, a_2, a_3). \quad (4.3.2)$$

The eigenvalues  $a_i$  satisfy the relations

$$a_i^3 - (a_1 + a_2 + a_3)a_i^2 + (a_1a_2 + a_2a_3 + a_3a_1)a_i - a_1a_2a_3 = 0, \quad (4.3.3)$$

i.e.

$$a_i^3 - \text{Tr}A \cdot a_i^2 + \frac{1}{2} \left[ (\text{Tr}A)^2 - \text{Tr}(A^2) \right] \cdot a_i - \det A = 0, \quad (4.3.4)$$

because

$$\text{Tr}A = a_1 + a_2 + a_3, \quad (4.3.5)$$

$$\text{csi}A \equiv \frac{1}{2} \left[ (\text{Tr}A)^2 - \text{Tr}(A^2) \right] = a_1a_2 + a_2a_3 + a_3a_1, \quad (4.3.6)$$

$$\det A = a_1a_2a_3. \quad (4.3.7)$$

(The function  $\text{csi}A$  has, for example, been used by Laboura<sup>3</sup>.) The relation (4.3.4) means

$$D^3 - \text{Tr}A \cdot D^2 + \frac{1}{2} \left[ (\text{Tr}A)^2 - \text{Tr}(A^2) \right] D - \det A = 0, \quad (4.3.8)$$

so that we obtain the relation (4.3.1).

**[Corollary]**

When we denote

$$c_1 = \text{Tr}A, \quad (4.3.9)$$

$$c_2 = \frac{1}{2} \left[ (\text{Tr}A)^2 - \text{Tr}(A^2) \right], \quad (4.3.10)$$

$$c_3 = \det A, \quad (4.3.11)$$

we can denote  $\text{Tr}(A^n)$  as follows:

$$\text{Tr}A = c_1, \quad (4.3.12)$$

$$\text{Tr}(A^2) = c_1^2 - 2c_2, \quad (4.3.13)$$

$$\begin{aligned} \text{Tr}(A^3) &= c_1 \text{Tr}(A^2) - c_2 \text{Tr}A + 3c_3 \\ &= c_1^3 - 3c_1c_2 + 3c_3, \end{aligned} \quad (4.3.14)$$

$$\begin{aligned} \text{Tr}(A^4) &= c_1 \text{Tr}(A^3) - c_2 \text{Tr}(A^2) + c_3 \text{Tr}A \\ &= c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2, \end{aligned} \quad (4.3.15)$$

$$\text{Tr}(A^5) = c_1^5 - 5c_1^3c_2 + 5c_1^2c_3 + 5c_1c_2^2 - 5c_2c_3 \quad (4.3.16)$$

$$\text{Tr}(A^6) = c_1^6 - 6c_1^4c_2 + 6c_1^3c_3 + 9c_1^2c_2^2 - 12c_1c_2c_3 - 2c_2^3 + 3c_3^2, \quad (4.3.17)$$

and so on, by using

$$\text{Tr}(A^{n+3}) - c_1 \text{Tr}(A^{n+2}) + c_2 \text{Tr}(A^{n+1}) - c_3 \text{Tr}(A^n) = 0. \quad (4.3.18)$$

**[Corollary]**

The expressions in terms of

$$\begin{aligned} t_1 &= \text{Tr}A, \\ t_2 &= \text{Tr}(AA), \\ t_3 &= \text{Tr}(AAA), \end{aligned} \quad (4.3.19)$$

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<sup>3</sup>L. Laboura, Phys. Rev. **D41**, 2275 (1990).

are also useful:

$$c_2 = \frac{1}{2}t_1^2 - \frac{1}{2}t_2, \quad (4.3.20)$$

$$c_3 \equiv \det A = \frac{1}{6}t_1^3 - \frac{1}{2}t_1t_2 + \frac{1}{3}t_3, \quad (4.3.21)$$

$$t_4 \equiv \text{Tr}(AAAA) = \frac{1}{6}t_1^4 - t_1^2t_2 + \frac{1}{2}t_2^2 + \frac{4}{3}t_1t_3, \quad (4.3.22)$$

$$A^3 - t_1A^2 + \frac{1}{2}(t_1^2 - t_2)A - \frac{1}{6}(t_1^3 - 3t_1t_2 + 2t_3) = 0, \quad (4.3.23)$$

$$A^{3+n} - t_1A^{2+n} + \frac{1}{2}(t_1^2 - t_2)A^{1+n} - \frac{1}{6}(t_1^3 - 3t_1t_2 + 2t_3)A^n = 0. \quad (4.3.24)$$

#### 4.4 Unitary matrix

An arbitrary  $3 \times 3$  unitary matrix  $U$  satisfies the relations

$$U_{il}U_{jm} - U_{im}U_{jl} = U_{kn}^* = (U^\dagger)_{nk}, \quad (4.4.1)$$

$$U_{il}U_{im}U_{jl}^*U_{jm}^* = \frac{1}{2} \left[ (U_{ik}U_{jk}^*)^2 - (U_{il}U_{jl}^*)^2 - (U_{im}U_{jm}^*)^2 \right], \quad (4.4.2)$$

$$\text{Re}(U_{il}U_{jm}U_{im}^*U_{jl}) = \frac{1}{2} \left( |U_{in}|^2|U_{jn}|^2 - |U_{il}|^2|U_{jl}|^2 - |U_{im}|^2|U_{jm}|^2 \right), \quad (4.4.3)$$

where  $(i, j, k)$  and  $(l, m, n)$  are cyclic permutations of  $(1, 2, 3)$ . The formula (4.4.1) explicitly means

$$U_{12}U_{23} - U_{13}U_{22} = U_{31}^*, \quad (4.4.4)$$

$$U_{13}U_{21} - U_{11}U_{23} = U_{32}^*, \quad (4.4.5)$$

and so on.

#### 4.5 Derivatives

Derivatives of a scalar function  $f(A)$  with respect to a  $3 \times 3$  matrix  $A$  are given as follows:

$$\frac{\partial}{\partial A} \text{Tr} A = \mathbf{1}, \quad (4.5.1)$$

$$\frac{\partial}{\partial A} \text{Tr}(AA) = 2A, \quad (4.5.2)$$

$$\frac{\partial}{\partial A} \text{Tr}(AAA) = 3AA, \quad (4.5.3)$$

$$\frac{\partial}{\partial A} \text{Tr}(BA) = B, \quad (4.5.4)$$

$$\frac{\partial}{\partial A} \text{Tr}(BAA) = BA + AB, \quad (4.5.5)$$

$$\frac{\partial}{\partial A} \text{Tr}(BAAA) = BAA + ABA + AAB, \quad (4.5.6)$$

and so on, where

$$\frac{\partial}{\partial A} f(A) = G, \quad (4.5.7)$$

denotes

$$\frac{\partial}{\partial A_{ab}} f(A) = G_{ba}. \quad (4.5.8)$$

Since derivative of a matrix  $GF$  with respect to  $A$  is given by

$$\frac{\partial(GF)_{ij}}{\partial A_{ab}} = \sum_k \frac{\partial G_{ik}}{\partial A_{ab}} F_{kj} + \sum_k G_{ik} \frac{\partial F_{kj}}{\partial A_{ab}}, \quad (4.5.9)$$

we obtain the following formulae:

$$\frac{\partial}{\partial A} A = \mathbf{1}, \quad (4.5.10)$$

$$\frac{\partial}{\partial A} AA = \mathbf{1} \otimes A + A \otimes \mathbf{1}, \quad (4.5.11)$$

$$\frac{\partial}{\partial A} AAA = \mathbf{1} \otimes AA + A \otimes A + A \otimes \mathbf{1}, \quad (4.5.12)$$

$$\frac{\partial}{\partial A} BA = B \otimes A, \quad (4.5.13)$$

$$\frac{\partial}{\partial A} BAA = B \otimes A + BA \otimes \mathbf{1}, \quad (4.5.14)$$

$$\frac{\partial}{\partial A} BAAA = B \otimes A + BA \otimes A + BAA \otimes \mathbf{1}, \quad (4.5.15)$$

and so on, where, for example, the formula (4.5.11) denotes

$$\frac{\partial}{\partial A_{ab}} (AA)_{ij} = \delta_{ia} A_{bj} + A_{ia} \delta_{bj}. \quad (4.5.16)$$

#### 4.6 $S = \varepsilon_{ikm} \varepsilon_{jln} A_{ij} B_{kl} C_{mn}$

In some models, a cubic term

$$S = \varepsilon_{ikm} \varepsilon_{jln} A_{ij} B_{kl} C_{mn}, \quad (4.6.1)$$

becomes important. For example, if  $A$ ,  $B$  and  $C$  are octets of an  $SU(3)$  symmetry, the matrix  $S$  is a singlet of the  $SU(3)$ . In an  $SO(3)$ ,  $S$  is also a singlet of the symmetry.

It is useful to express the matrix  $S$  in terms of traces of  $A$ ,  $B$  and  $C$ . The matrix  $S$  can be expressed as follows:

$$S = \text{Tr}[ABC + CBA] - \text{Tr}[A]\text{Tr}[BC] - \text{Tr}[B]\text{Tr}[CA] - \text{Tr}[C]\text{Tr}[AB] + \text{Tr}[A]\text{Tr}[B]\text{Tr}[C]. \quad (4.6.2)$$

The quantity  $S$  is totally symmetric under an exchange among  $A$ ,  $B$  and  $C$ .

**[Proof]**

We can write  $S$  explicitly as

$$S = \sum_{(1,2,3)} \sum_{(i,j,k)} [(A_{i1} B_{k2} - A_{i2} B_{k1}) - (A_{k1} B_{i2} - A_{k2} B_{i1})] C_{m3}, \quad (4.6.3)$$

where  $(i, j, k)$  denotes cyclic permutation of  $(1, 2, 3)$ , and  $\sum_{(1,2,3)}$  denotes  $\sum_{(1,2,3)} X_{12}a_3 = X_{12}a_3 + X_{23}a_1 + X_{31}a_2$ . For example, the coefficient of  $C_{23}$  is given by

$$\begin{aligned}
& A_{31}B_{12} + A_{12}B_{31} - A_{32}B_{11} - A_{11}B_{32} \\
&= A_{31}B_{12} + A_{12}B_{31} - A_{32}B_{11} - A_{11}B_{32} \\
&+ A_{32}B_{22} + A_{22}B_{32} - A_{32}B_{22} - A_{22}B_{32} \\
&+ A_{33}B_{32} + A_{32}B_{33} - A_{32}B_{33} - A_{33}B_{32} \\
&= (AB)_{32} + (BA)_{32} - A_{32}\text{Tr}[B] - \text{Tr}[A]B_{32}.
\end{aligned} \tag{4.6.3}$$

Also, we can find that the coefficient of  $C_{33}$

$$A_{11}B_{22} + A_{22}B_{11} - A_{12}B_{21} - A_{21}B_{12}, \tag{4.6.4}$$

can be written as

$$(AB)_{33} + (BA)_{33} - A_{33}\text{Tr}[B] - \text{Tr}[A]B_{33} + \text{Tr}[A]\text{Tr}[B] - \text{Tr}[AB]. \tag{4.6.5}$$

Therefore, in generally, the coefficient of  $C_{ij}$  can be expressed as

$$(AB)_{ji} + (BA)_{ji} - A_{ji}\text{Tr}[B] - \text{Tr}[A]B_{ji} + \delta_{ji} (\text{Tr}[A]\text{Tr}[B] - \text{Tr}[AB]), \tag{4.6.6}$$

so that we can obtain the formula (4.6.2).

**[Corollary]**

For  $B = C$ , we obtain

$$S = 2\text{Tr}[ABB] - \text{Tr}[A]\text{Tr}[BB] - 2\text{Tr}[B]\text{Tr}[BA] + \text{Tr}[A]\text{Tr}^2[B]. \tag{4.6.7}$$

Also, for  $A = B = C$ , we obtain

$$S = 2\text{Tr}[AAA] - 3\text{Tr}[A]\text{Tr}[AA] + \text{Tr}^3[A]. \tag{4.6.8}$$

By comparing with (4.1.1), we find

$$S = 6 \det A. \tag{4.6.9}$$

## 5 Eigenvalues of an Arbitrary $3 \times 3$ Matrix

General formulae for eigenvalues of an arbitrary  $3 \times 3$  Hermitian matrix are given. The formulae are useful for investigating the mass spectrum of the  $3 \times 3$  mass matrix.

### 5.1 General case

For convenience, we denote the case that the matrix  $M$  is Hermitian. If it is not so, we can make a Hermitian matrix by  $MM^\dagger$  (or  $M^\dagger M$ ).

Generally, the eigenvalues  $x_i$  ( $i = 1, 2, 3$ ) are given by solutions of the cubic equation

$$x^3 - ax^2 + bx - c = 0, \quad (5.1.1)$$

where

$$a = \text{Tr}M, \quad (5.1.2)$$

$$b = \frac{1}{2} [(\text{Tr}M)^2 - \text{Tr}M^2], \quad (5.1.3)$$

$$c = \det M. \quad (5.1.4)$$

Since  $M$  is a  $3 \times 3$  matrix, we can rewrite  $\det M$  as

$$c = \frac{1}{3} \text{Tr}M^3 - \frac{1}{2} \text{Tr}M \text{Tr}M^2 + \frac{1}{6} (\text{Tr}M)^3. \quad (5.1.5)$$

When we define

$$y = x - \frac{1}{3}a, \quad (5.1.6)$$

we obtain a cubic equation

$$y^3 - 3py - q = 0, \quad (5.1.7)$$

where

$$p = \frac{1}{9}(a^2 - 3b), \quad (5.1.8)$$

$$q = \frac{1}{27}(2a^3 - 9ab + 27c). \quad (5.1.9)$$

Then, from the Cardano's formula, we obtain the following solutions  $y_i$  ( $i = 1, 2, 3$ )

$$\begin{aligned} y_1 &= \alpha + \beta, \\ y_2 &= \omega\alpha + \omega^2\beta, \\ y_3 &= \omega^2\alpha + \omega\beta, \end{aligned} \quad (5.1.10)$$

where

$$\omega = e^{i\frac{2}{3}\pi} = \frac{-1 + i\sqrt{3}}{2}, \quad (5.1.11)$$

$$\alpha = \left( \frac{q + \sqrt{q^2 - 4p^3}}{2} \right)^{\frac{1}{3}}, \quad \beta = \left( \frac{q - \sqrt{q^2 - 4p^3}}{2} \right)^{\frac{1}{3}}. \quad (5.1.12)$$

Therefore, we obtain the following solutions  $x_i (= \lambda_i)$

$$\begin{aligned}x_1 &= \frac{1}{3}a + \alpha + \beta, \\x_2 &= \frac{1}{3}a + \omega\alpha + \omega^2\beta, \\x_3 &= \frac{1}{3}a + \omega^2\alpha + \omega\beta,\end{aligned}\tag{5.1.13}$$

Note that

$$y_1 + y_2 + y_3 = 0,\tag{5.1.14}$$

so that

$$x_1 + x_2 + x_3 = a.\tag{5.1.15}$$

The result (4.1.14) is a matter of course, because

$$a = \text{Tr}M = \lambda_1 + \lambda_2 + \lambda_3.\tag{5.1.16}$$

## 5.2 Real eigenvalues

From the definitions of  $\alpha$  and  $\beta$ , (4.1.12), we find that we can obtain three real solutions  $y_i (x_i)$  only for the case with

$$p > 0 \quad \text{and} \quad 4p^3 - q^2 > 0.\tag{5.2.1}$$

Under the condition (5.2.1), we can write

$$\begin{aligned}\alpha^3 &= \frac{1}{2} \left( q + i\sqrt{4p^3 - q^2} \right) = r^3 e^{i3\theta}, \\ \beta^3 &= \frac{1}{2} \left( q - i\sqrt{4p^3 - q^2} \right) = r^3 e^{-i3\theta},\end{aligned}\tag{5.2.2}$$

where

$$r^6 = \alpha^3 \beta^3 = \frac{1}{4} [q^2 - (q^2 - 4p^3)] = p^3,\tag{5.2.3}$$

i.e.

$$r = \sqrt{p},\tag{5.2.4}$$

and

$$\tan 3\theta = \frac{\text{Im} \alpha^3}{\text{Re} \alpha^3} = \frac{\sqrt{4p^3 - q^2}}{q},\tag{5.2.5}$$

i.e.

$$\cos 3\theta = \pm \frac{1}{2r^3} q, \quad \sin 3\theta = \pm \frac{1}{2r^3} \sqrt{4p^3 - q^2}.\tag{5.2.6}$$

Therefore, we obtain

$$\begin{aligned}y_1 &= \alpha + \beta = 2r \cos \theta, \\ y_2 &= \omega\alpha + \omega^2\beta = 2r \cos \left( \theta + \frac{2}{3}\pi \right), \\ y_3 &= \omega^2\alpha + \omega\beta = 2r \cos \left( \theta - \frac{2}{3}\pi \right).\end{aligned}\tag{5.2.7}$$

From Eqs. (5.1.2)–(5.1.4), we can denote

$$p = \frac{1}{9}a^2 - \frac{1}{3}b = \frac{1}{18} [3\text{Tr}M^2 - (\text{Tr}M)^2],\tag{5.2.8}$$

i.e.

$$r = \frac{1}{3\sqrt{2}}\sqrt{3\text{Tr}M^2 - (\text{Tr}M)^2}, \quad (5.2.9)$$

and

$$q = \frac{2}{27}a^3 - \frac{1}{3}ab + c = -\frac{5}{54}(\text{Tr}M)^3 + \frac{9}{54}\text{Tr}M\text{Tr}M^2 + \det M. \quad (5.2.10)$$

In conclusion, we obtain the eigenvalues  $x_i$  of  $M$

$$\begin{aligned} x_1 &= \frac{1}{3}a + 2r \cos \theta, \\ x_2 &= \frac{1}{3}a + 2r \cos(\theta + \frac{2}{3}\pi), \\ x_3 &= \frac{1}{3}a + 2r \cos(\theta - \frac{2}{3}\pi). \end{aligned} \quad (5.2.11)$$

Note that

$$\det M = x_1 x_2 x_3 = \frac{1}{27}a^3 + ar^2 + 2r^3 \cos 3\theta. \quad (5.2.12)$$

The relation (5.2.12) will be useful for evaluation of the value of  $\theta$ .

### 5.3 Formulae for 3 eigenvalues

As we discussed in Sec.4.2, especially, in Eq.(5.2.11), it is useful to express the 3 eigenvalues as

$$\begin{aligned} x_1 &= \frac{a}{\sqrt{6}} - \frac{b}{\sqrt{3}} \sin \theta, \\ x_2 &= \frac{a}{\sqrt{6}} - \frac{b}{\sqrt{3}} \sin(\theta + \frac{2}{3}\pi), \\ x_3 &= \frac{a}{\sqrt{6}} - \frac{b}{\sqrt{3}} \sin(\theta + \frac{4}{3}\pi). \end{aligned} \quad (5.3.1)$$

Here, we have redefined the parameters  $a$  and  $\theta$  against Eq.(5.2.11). From the expression (5.3.1), we obtain the following relations:

$$\text{Tr}M = x_1 + x_2 + x_3 = \sqrt{\frac{3}{2}}a, \quad (5.3.2)$$

$$\text{Tr}M^2 = x_1^2 + x_2^2 + x_3^2 = \frac{1}{2}(a^2 + b^2), \quad (5.3.3)$$

$$\frac{1}{2}[(\text{Tr}M)^2 - \text{Tr}M^2] = x_1 x_2 + x_2 x_3 + x_3 x_1 = \frac{1}{4}(2a^2 - b^2), \quad (5.3.4)$$

$$\det M = x_1 x_2 x_3 = \frac{1}{24\sqrt{3}}[\sqrt{2}a(2a^2 - 3b^2) + 2b^3 \sin 3\theta], \quad (5.3.5)$$

$$\text{Tr}M^4 = x_1^4 + x_2^4 + x_3^4 = \frac{1}{24}(2a^4 + 12a^2b^2 + 3b^4 + 4\sqrt{2}ab^3 \sin 3\theta). \quad (5.3.6)$$

Note that the quantities  $\Sigma x_i$  and  $\Sigma x_i^2$  are independent of the parameter  $\theta$ .

## 6 Matrix Form Under a Specific Rotation

We give matrix forms which are transformed by a specific 3-dimensional rotation.

### 6.1 Rotation in a two-dimensional plain

We define the following two-dimensional rotations

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix}, \quad (6.1.1)$$

$$R_2 = \begin{pmatrix} c & 0 & s \\ 0 & 1 & 0 \\ -s & 0 & c \end{pmatrix}, \quad (6.1.2)$$

$$R_3 = \begin{pmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (6.1.3)$$

where  $s = \sin \theta$  and  $c = \cos \theta$ . Then, the matrix  $M$

$$M = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \quad (6.1.4)$$

is transformed as follows:

$$R_1^T M R_1 = \begin{pmatrix} M_{11} & M_{12}c - M_{13}s & M_{13}c + M_{12}s \\ M_{21}c - M_{31}s & M_{22}c^2 + M_{33}s^2 - (M_{23} + M_{32})cs & (M_{22} - M_{33})cs + M_{23}c^2 - M_{32}s^2 \\ M_{31}c + M_{21}s & (M_{22} - M_{33})cs + M_{32}c^2 - M_{23}s^2 & M_{22}s^2 + M_{33}c^2 + (M_{23} + M_{32})cs \end{pmatrix}, \quad (6.1.5)$$

$$R_2^T M R_2 = \begin{pmatrix} M_{11}c^2 + M_{33}s^2 - (M_{13} + M_{31})cs & M_{12}c - M_{32}s & M_{13}c^2 - M_{31}s^2 + (M_{11} - M_{33})cs \\ M_{21}c - M_{23}s & M_{22} & M_{23}c + M_{21}s \\ M_{31}c^2 - M_{13}s^2 + (M_{11} - M_{33})cs & M_{32}c + M_{12}s & M_{11}s^2 + M_{33}c^2 + (M_{13} + M_{31})cs \end{pmatrix}, \quad (6.1.6)$$

$$R_3^T M R_3 = \begin{pmatrix} M_{11}c^2 + M_{22}s^2 - (M_{12} + M_{21})cs & M_{12}c^2 - M_{21}s^2 + (M_{11} - M_{22})cs & M_{13}c - M_{23}s \\ M_{21}c^2 - M_{12}s^2 + (M_{11} - M_{22})cs & M_{11}s^2 + M_{22}c^2 + (M_{12} + M_{21})cs & M_{23}c + M_{13}s \\ M_{31}c - M_{32}s & M_{32}c + M_{31}s & M_{33} \end{pmatrix}. \quad (6.1.7)$$

## 6.2 Rotation into an $S_3$ basis

Under the rotation

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{\frac{1}{3}} & -\sqrt{\frac{2}{3}} \\ 0 & \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} \end{pmatrix} \begin{pmatrix} \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} & 0 \\ \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}, \quad (6.2.1)$$

matrices  $M$  are transformed as shown in Table 6.2.1.

**Table 6.2.1** Rotation into an  $S_3$  basis

$M$	$UMU^T$	$U^T MU$	Eq. No.
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\frac{1}{6} \begin{pmatrix} 3 & \sqrt{3} & \sqrt{6} \\ \sqrt{3} & 1 & \sqrt{2} \\ \sqrt{6} & \sqrt{2} & 2 \end{pmatrix}$	$\frac{1}{2} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	(6.2.2)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\frac{1}{6} \begin{pmatrix} 3 & -\sqrt{3} & -\sqrt{6} \\ -\sqrt{3} & 1 & \sqrt{2} \\ -\sqrt{6} & \sqrt{2} & 2 \end{pmatrix}$	$\frac{1}{6} \begin{pmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ -2 & -2 & 4 \end{pmatrix}$	(6.2.3)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\frac{1}{3} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & -\sqrt{2} \\ 0 & -\sqrt{2} & 1 \end{pmatrix}$	$\frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	(6.2.4)
$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\frac{1}{3} \begin{pmatrix} -3 & 0 & 0 \\ 0 & 1 & \sqrt{2} \\ 0 & \sqrt{2} & 2 \end{pmatrix}$	$\frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$	(6.2.5)
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\frac{1}{6} \begin{pmatrix} 0 & -2\sqrt{3} & \sqrt{6} \\ -2\sqrt{3} & -4 & -\sqrt{2} \\ \sqrt{6} & -\sqrt{2} & 4 \end{pmatrix}$	$\frac{1}{\sqrt{6}} \begin{pmatrix} 2 & 0 & 1 \\ 0 & -2 & -1 \\ 1 & -1 & 0 \end{pmatrix}$	(6.2.6)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$\frac{1}{6} \begin{pmatrix} 0 & 2\sqrt{3} & -\sqrt{6} \\ 2\sqrt{3} & -4 & -\sqrt{2} \\ -\sqrt{6} & -\sqrt{2} & 4 \end{pmatrix}$	$\frac{1}{3\sqrt{2}} \begin{pmatrix} 2 & 2 & -1 \\ 2 & 2 & -1 \\ -1 & -1 & -4 \end{pmatrix}$	(6.2.7)
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\frac{1}{3} \begin{pmatrix} 0 & \sqrt{3} & \sqrt{6} \\ \sqrt{3} & 0 & 0 \\ \sqrt{6} & 0 & 0 \end{pmatrix}$	$\frac{1}{3} \begin{pmatrix} 1 & -2 & 1 \\ -2 & 1 & 1 \\ 1 & 1 & -2 \end{pmatrix}$	(6.2.8)
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}$	$-\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$	(6.2.9)

$M$	$UMU^T$	$U^T MU$	Eq. No.
$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	$3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$		(6.2.10)
$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\frac{1}{3} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & \sqrt{2} \\ 0 & \sqrt{2} & 5 \end{pmatrix}$	$\frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{3}+1 & 0 & -1 \\ 0 & \sqrt{3}-1 & 1 \\ -1 & 1 & \sqrt{3} \end{pmatrix}$	(6.2.11)
$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$	$\frac{1}{3} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 8 & 2\sqrt{2} \\ 0 & 2\sqrt{2} & 1 \end{pmatrix}$		(6.2.12)
$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$	$\frac{\sqrt{3}}{3} \begin{pmatrix} 0 & 2 & -\sqrt{2} \\ 2 & 0 & 0 \\ -\sqrt{2} & 0 & 0 \end{pmatrix}$		(6.2.13)
$\begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$	$\sqrt{3} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$		(6.2.14)
$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 1 & \sqrt{2} \\ -1 & 0 & 0 \\ -\sqrt{2} & 0 & 0 \end{pmatrix}$	$\frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$	(6.2.15)
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$	$\frac{1}{\sqrt{6}} \begin{pmatrix} 0 & -\sqrt{2} & 1 \\ \sqrt{2} & 0 & \sqrt{3} \\ -1 & -\sqrt{3} & 0 \end{pmatrix}$	$\frac{1}{\sqrt{6}} \begin{pmatrix} 0 & 2 & 1 \\ -2 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$	(6.2.16)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$	$\frac{1}{\sqrt{6}} \begin{pmatrix} 0 & \sqrt{2} & -1 \\ -\sqrt{2} & 0 & \sqrt{3} \\ 1 & -\sqrt{3} & 0 \end{pmatrix}$	$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$	(6.2.17)

### 6.3 Bi-maximal mixing

Under the bi-maximal mixing

$$U = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} = U^T, \quad (6.3.1)$$

matrices  $M$  are transformed as shown in Table 6.3.1.

**Table 6.3.1** Rotation under the bi-maximal mixing

$M$	$UMU^T = U^T M U$	Eq. No.
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\frac{1}{4} \begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 2 \end{pmatrix}$	(6.3.2)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\frac{1}{4} \begin{pmatrix} 1 & 1 & -\sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ -\sqrt{2} & -\sqrt{2} & 2 \end{pmatrix}$	(6.3.3)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\frac{1}{2} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	(6.3.4)
$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$	(6.3.5)
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\frac{1}{2} \begin{pmatrix} \sqrt{2} & 0 & 1 \\ 0 & -\sqrt{2} & -1 \\ 1 & -1 & 0 \end{pmatrix}$	(6.3.6)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$\frac{1}{2} \begin{pmatrix} \sqrt{2} & 0 & -1 \\ 0 & -\sqrt{2} & 1 \\ -1 & 1 & 0 \end{pmatrix}$	(6.3.7)
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$	(6.3.8)
$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}$	(6.3.9)

$M$	$UMU^T = U^T MU$	Eq. No.
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$	$\frac{1}{2} \begin{pmatrix} 0 & -\sqrt{2} & -1 \\ \sqrt{2} & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$	(6.3.10)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$	$\frac{1}{2} \begin{pmatrix} 0 & -\sqrt{2} & 1 \\ \sqrt{2} & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$	(6.3.11)
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$	(6.3.12)
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$	(6.3.13)
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$	$\frac{1}{2} \begin{pmatrix} -1 & 3 & 0 \\ 3 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$	(6.3.14)
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$	$\sqrt{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	(6.3.15)
$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	(6.3.16)
$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	$\frac{1}{2} \begin{pmatrix} 3 + 2\sqrt{2} & 1 & 0 \\ 1 & 3 - 2\sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	(6.3.17)
$\begin{pmatrix} 1 & 1 & i \\ 1 & 1 & i \\ -i & -i & 1 \end{pmatrix}$	$\frac{1}{2} \begin{pmatrix} 3 & 1 - 2\sqrt{2}i & 0 \\ 1 + 2\sqrt{2}i & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	(6.3.18)
$\begin{pmatrix} 0 & 0 & e^{i\phi} \\ 0 & 0 & e^{i\phi} \\ e^{-i\phi} & e^{-i\phi} & 0 \end{pmatrix}$	$\sqrt{2} \begin{pmatrix} \cos \phi & -i \sin \phi & 0 \\ i \sin \phi & -\cos \phi & 0 \\ 0 & 0 & 0 \end{pmatrix}$	(6.3.19)
$\begin{pmatrix} 0 & 0 & e^{i\phi} \\ 0 & 0 & e^{-i\phi} \\ e^{-i\phi} & e^{i\phi} & 0 \end{pmatrix}$	$\begin{pmatrix} \sqrt{2} \cos \phi & 0 & -i \sin \phi \\ 0 & -\sqrt{2} \cos \phi & i \sin \phi \\ i \sin \phi & -i \sin \phi & 0 \end{pmatrix}$	(6.3.20)

## 6.4 Formulae on the tri-maximal mixing

### 6.4.1 Definition

The tri-maximal mixing matrix  $V_T$  is defined by

$$V_T = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ \omega & \omega^2 & 1 \\ \omega^2 & \omega & 1 \end{pmatrix}, \quad V_T^\dagger = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \\ 1 & 1 & 1 \end{pmatrix}, \quad (6.4.1)$$

where

$$\omega = e^{i\frac{2}{3}\pi} = \frac{-1 + i\sqrt{3}}{2}, \quad (6.4.2)$$

$$\omega^3 = 1, \quad (6.4.3)$$

$$1 + \omega + \omega^2 = 0. \quad (6.4.4)$$

The matrix  $V_T$  satisfies the following relations

$$(V_T)^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \omega & 0 \\ \omega^2 & 0 & 0 \end{pmatrix}, \quad (6.4.5)$$

$$(V_T)^4 = \omega^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (6.4.6)$$

### 6.4.2 Transformation $V_T M V_T^\dagger$

The matrix forms under the transformation  $V_T M V_T^\dagger$  are given as follows:

$$V_T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V_T^\dagger = \frac{1}{3} \begin{pmatrix} 1 & \omega^2 & \omega \\ \omega & 1 & \omega^2 \\ \omega^2 & \omega & 1 \end{pmatrix}, \quad (6.4.7)$$

$$V_T \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} V_T^\dagger = \frac{1}{3} \begin{pmatrix} 1 & \omega & \omega^2 \\ \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \end{pmatrix}, \quad (6.4.8)$$

$$V_T \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} V_T^\dagger = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad (6.4.9)$$

$$V_T \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V_T^\dagger = \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & -1 & 2 \\ -1 & 2 & -1 \end{pmatrix}, \quad (6.4.10)$$

$$V_T \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} V_T^\dagger = \frac{1}{3} \begin{pmatrix} 2 & -\omega & -\omega^2 \\ -\omega^2 & -1 & 2\omega \\ -\omega & 2\omega^2 & -1 \end{pmatrix}, \quad (6.4.11)$$

$$V_T \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} V_T^\dagger = \frac{1}{3} \begin{pmatrix} 2 & -\omega^2 & -\omega \\ -\omega & -1 & 2\omega^2 \\ -\omega^2 & 2\omega & -1 \end{pmatrix}, \quad (6.4.12)$$

$$V_T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} V_T^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \omega^2 \\ 0 & \omega & 0 \end{pmatrix}, \quad (6.4.13)$$

$$V_T \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} V_T^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \omega \\ 0 & \omega^2 & 0 \end{pmatrix}, \quad (6.4.14)$$

$$V_T \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} V_T^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (6.4.15)$$

$$V_T \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V_T^\dagger = \frac{1}{3} \begin{pmatrix} 0 & \omega - \omega^2 & \omega^2 - \omega \\ \omega - \omega^2 & \omega^2 - \omega & 0 \\ \omega^2 - \omega & 0 & \omega - \omega^2 \end{pmatrix}, \quad (6.4.16)$$

$$V_T \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} V_T^\dagger = \frac{1}{3} \begin{pmatrix} 0 & \omega^2 - 1 & \omega - 1 \\ 1 - \omega & \omega^2 - \omega & 0 \\ 1 - \omega^2 & 0 & \omega - \omega^2 \end{pmatrix}, \quad (6.4.17)$$

$$V_T \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} V_T^\dagger = \frac{1}{3} \begin{pmatrix} 0 & 1 - \omega & 1 - \omega^2 \\ \omega^2 - 1 & \omega^2 - \omega & 0 \\ \omega - 1 & 0 & \omega - \omega^2 \end{pmatrix}, \quad (6.4.18)$$

$$V_T \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} V_T^\dagger = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad (6.4.19)$$

$$V_T \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} V_T^\dagger = (\omega^2 - \omega) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad (6.4.20)$$

$$V_T \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} V_T^\dagger = \omega^2 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad (6.4.21)$$

$$V_T \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix} V_T^\dagger = \omega \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \quad (6.4.22)$$

### 6.4.3 Transformation $V_T^\dagger M V_T$

The matrix forms under the transformation  $V_T^\dagger M V_T$  are given as follows:

$$V_T^\dagger \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V_T = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad (6.4.23)$$

$$V_T^\dagger \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} V_T = \frac{1}{3} \begin{pmatrix} 1 & \omega & \omega^2 \\ \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \end{pmatrix}, \quad (6.4.24)$$

$$V_T^\dagger \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} V_T = \frac{1}{3} \begin{pmatrix} 1 & \omega^2 & \omega \\ \omega & 1 & \omega^2 \\ \omega^2 & \omega & 1 \end{pmatrix}, \quad (6.4.25)$$

$$V_T^\dagger \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V_T = \frac{1}{3} \begin{pmatrix} -1 & 2\omega^2 & -\omega \\ 2\omega & -1 & -\omega^2 \\ -\omega^2 & -\omega & 2 \end{pmatrix}, \quad (6.4.26)$$

$$V_T^\dagger \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} V_T = \frac{1}{3} \begin{pmatrix} -1 & 2\omega & -\omega^2 \\ 2\omega^2 & -1 & -\omega \\ -\omega & -\omega^2 & 2 \end{pmatrix}, \quad (6.4.27)$$

$$V_T^\dagger \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} V_T = \frac{1}{3} \begin{pmatrix} -1 & 2 & -1 \\ 2 & -1 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \quad (6.4.28)$$

$$V_T^\dagger \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V_T = \frac{1}{3} \begin{pmatrix} \omega - \omega^2 & 0 & 1 - \omega^2 \\ 0 & \omega^2 - \omega & 1 - \omega \\ \omega - 1 & \omega^2 - 1 & 0 \end{pmatrix}, \quad (6.4.29)$$

$$V_T^\dagger \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} V_T = \frac{1}{3} \begin{pmatrix} \omega - \omega^2 & 0 & \omega - 1 \\ 0 & \omega^2 - \omega & \omega^2 - 1 \\ 1 - \omega^2 & 1 - \omega & 0 \end{pmatrix}, \quad (6.4.30)$$

$$V_T^\dagger \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} V_T = \frac{1}{3} \begin{pmatrix} \omega - \omega^2 & 0 & \omega^2 - \omega \\ 0 & \omega^2 - \omega & \omega - \omega^2 \\ \omega^2 - \omega & \omega - \omega^2 & 0 \end{pmatrix}, \quad (6.4.31)$$

$$V_T^\dagger \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} V_T = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad (6.4.32)$$

$$V_T^\dagger \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} V_T = (\omega - \omega^2) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (6.4.33)$$

$$V_T^\dagger \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} V_T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \quad (6.4.34)$$

#### 6.4.4 Transformation $V_T M V_T^T$

The matrix forms under the transformation  $V_T M V_T^T$  are given as follows:

$$V_T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} V_T^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (6.4.35)$$

$$V_T \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} V_T^T = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad (6.4.36)$$

$$V_T \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} V_T^T = \omega^2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad (6.4.37)$$

$$V_T \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix} V_T^T = \omega \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (6.4.38)$$

$$V_T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} V_T^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \quad (6.4.39)$$

$$V_T \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} V_T^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad (6.4.40)$$

$$V_T \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} V_T^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (6.4.41)$$

$$V_T \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} V_T^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \omega^2 \\ 0 & \omega & 0 \end{pmatrix}, \quad (6.4.42)$$

$$V_T \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} V_T^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \omega \\ 0 & \omega^2 & 0 \end{pmatrix}, \quad (6.4.43)$$

$$V_T \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} V_T^T = (\omega^2 - \omega) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad (6.4.44)$$

$$V_T \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} V_T^T = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \quad (6.4.45)$$

#### 6.4.5 Transformation $V_T^\dagger M V_T^*$

The matrix forms under the transformation  $V_T^\dagger M V_T^*$  are given as follows:

$$V_T^\dagger \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} V_T^* = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (6.4.46)$$

$$V_T^\dagger \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} V_T^* = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad (6.4.47)$$

$$V_T^\dagger \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} V_T^* = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad (6.4.48)$$

$$V_T^\dagger \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix} V_T^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (6.4.49)$$

$$V_T^\dagger \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} V_T^* = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (6.4.50)$$

$$V_T^\dagger \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} V_T^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (6.4.51)$$

#### 6.4.6 Transformation for a circulant matrix

When we define a circulant matrix  $T(\theta)$  as

$$T(\theta) = \begin{pmatrix} 0 & e^{i\theta} & e^{-i\theta} \\ e^{-i\theta} & 0 & e^{i\theta} \\ e^{i\theta} & e^{-i\theta} & 0 \end{pmatrix} = T^\dagger(\theta), \quad (6.4.52)$$

we obtain the following relations:

$$V_T T(\theta) V_T^\dagger = \cos \theta \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \sqrt{3} \sin \theta \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad (6.4.53)$$

$$V_T^\dagger T(\theta) V_T = \cos \theta \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} - \sqrt{3} \sin \theta \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (6.4.54)$$

$$V_T T(\theta) V_T^T = \cos \theta \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} + \sqrt{3} \sin \theta \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad (6.4.55)$$

$$T^2(\theta) = 2\mathbf{1} + T(-2\theta), \quad (6.4.56)$$

$$T(\alpha + \beta) = \cos \beta T(\alpha) + \sin \beta T\left(\alpha + \frac{\pi}{2}\right), \quad (6.4.57)$$

$$aT(\alpha) + bT(\beta) = \sqrt{a^2 + b^2 + 2ab \cos(\alpha - \beta)} T(\phi), \quad (6.4.58)$$

$$\tan \phi = \frac{a \sin \alpha + b \sin \beta}{a \cos \alpha + b \cos \beta}. \quad (6.4.59)$$

# 7 Democratic Mass Matrix

Formulae related to the so-called “democratic” mass matrix model are given.

## 7.1 General Formulae

A nearly democratic<sup>4</sup> mass matrix  $M$  is given by

$$M = aX + bY, \quad (7.1.1)$$

where  $|a|^2 \gg |b|^2$ ,  $X$  is the so-called “democratic” matrix

$$X = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad (7.1.2)$$

and  $Y$  is a  $3 \times 3$  matrix with the order of one. The democratic matrix  $X$  is diagonalized by the mixing matrix  $A$

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}, \quad (7.1.3)$$

as

$$AXA^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (7.1.4)$$

## 7.2 Inverse

The democratic matrix  $X$  defined by Eq.(7.1.2) does not have the inverse, because  $\det X = 0$ . However, when we define

$$X(\phi) = \frac{1}{3} \begin{pmatrix} 1 & e^{i\phi} & e^{-i\phi} \\ e^{-i\phi} & 1 & e^{i\phi} \\ e^{i\phi} & e^{-i\phi} & 1 \end{pmatrix}, \quad (7.2.1)$$

the matrix  $X(\phi)$  has the inverse

$$X(\phi)^{-1} = |b| \begin{pmatrix} 0 & e^{i\beta} & e^{-i\beta} \\ e^{-i\beta} & 0 & e^{i\beta} \\ e^{i\beta} & e^{-i\beta} & 0 \end{pmatrix}, \quad (7.2.2)$$

where

$$|b| = \frac{3}{2} \frac{1}{\sin(\frac{3}{2}\phi)}, \quad (7.2.3)$$

---

<sup>4</sup>The terminology “democratic” was first named by Jarlskog: C. Jarlskog, in Proceedings of the International Symposium on *Production and Decays of Heavy Hadrons*, Heidelberg, Germany, 1986 edited by K. R. Schubert.

$$\beta = \frac{1}{2}(\pi - \phi). \quad (7.2.4)$$

An approximately democratic mass matrix

$$M = \begin{pmatrix} 1 + \varepsilon & e^{i\phi} & e^{-i\phi} \\ e^{-i\phi} & 1 + \varepsilon & e^{i\phi} \\ e^{i\phi} & e^{-i\phi} & 1 + \varepsilon \end{pmatrix} = 3X(\phi) + \varepsilon \mathbf{1}, \quad (7.2.5)$$

has the inverse

$$M^{-1} = \begin{pmatrix} a & b & b^* \\ b^* & a & b \\ b & b^* & a \end{pmatrix} = |b| \left[ 3X(\beta) - \left( 1 - \frac{a}{|b|} \right) \mathbf{1} \right], \quad (7.2.6)$$

where

$$a = \frac{(2 + \varepsilon)\varepsilon}{4 \sin^2(\frac{3}{2}\phi) - (3 + \varepsilon)\varepsilon^2}, \quad (7.2.7)$$

$$b = |b|e^{i\beta}, \quad (7.2.8)$$

$$|b| = \frac{\sqrt{\varepsilon^2 \cos^2(\frac{3}{2}\phi) + (2 + \varepsilon)^2 \sin^2(\frac{3}{2}\phi)}}{4 \sin^2(\frac{3}{2}\phi) - (3 + \varepsilon)\varepsilon^2}, \quad (7.2.9)$$

and

$$\beta = \pi + \delta - \frac{1}{2}\phi, \quad (7.2.10)$$

$$\begin{aligned} \sin \delta &= \frac{(2 + \varepsilon) \sin(\frac{3}{2}\phi)}{\sqrt{\varepsilon^2 \cos^2(\frac{3}{2}\phi) + (2 + \varepsilon)^2 \sin^2(\frac{3}{2}\phi)}}, \\ \cos \delta &= \frac{\varepsilon \cos(\frac{3}{2}\phi)}{\sqrt{\varepsilon^2 \cos^2(\frac{3}{2}\phi) + (2 + \varepsilon)^2 \sin^2(\frac{3}{2}\phi)}}, \end{aligned} \quad (7.2.11)$$

$$\frac{a}{|b|} = \frac{(2 + \varepsilon)\varepsilon}{\sqrt{\varepsilon^2 \cos^2(\frac{3}{2}\phi) + (2 + \varepsilon)^2 \sin^2(\frac{3}{2}\phi)}}. \quad (7.2.12)$$

For a case with general phases

$$M = \begin{pmatrix} 1 + \varepsilon & e^{i\phi_{12}} & e^{-i\phi_{31}} \\ e^{-i\phi_{12}} & 1 + \varepsilon & e^{i\phi_{23}} \\ e^{i\phi_{31}} & e^{-i\phi_{23}} & 1 + \varepsilon \end{pmatrix}, \quad (7.2.13)$$

we define a phase matrix  $P$

$$P = \text{diag}(e^{i\delta_1}, e^{i\delta_2}, e^{i\delta_3}), \quad (7.2.14)$$

and we denote  $M$  as

$$M = P \left( 3X(\phi) + \varepsilon \mathbf{1} \right) P^\dagger, \quad (7.2.15)$$

where

$$\phi_{12} = \phi + \delta_1 - \delta_2,$$

$$\begin{aligned}\phi_{23} &= \phi + \delta_2 - \delta_3, \\ \phi_{31} &= \phi + \delta_3 - \delta_1,\end{aligned}\tag{7.2.16}$$

and

$$\phi_{12} + \phi_{23} + \phi_{31} = 3\phi.\tag{7.2.17}$$

Then, we obtain

$$M^{-1} = |b|P \left[ 3X(\beta) - \left( 1 - \frac{a}{|b|} \right) \mathbf{1} \right] P^\dagger.\tag{7.2.18}$$

### 7.3 Nearly democratic mass matrix

Let us diagonalize a nearly democratic mass matrix<sup>5</sup>

$$M = \frac{1}{3}a \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix}.\tag{7.3.1}$$

We define the following mixing matrix  $U_D$

$$U_D = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}.\tag{7.3.2}$$

Then, from the formulae (6.2.2) ~ (6.2.4) and (6.2.10) for the rotation by  $U_D$ , we obtain

$$\begin{aligned}M_D \equiv U_D^T M U_D &= a \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &+ \frac{1}{6} \begin{pmatrix} 3(\varepsilon_1 + \varepsilon_2) & \sqrt{3}(\varepsilon_1 - \varepsilon_2) & \sqrt{6}(\varepsilon_1 - \varepsilon_2) \\ \sqrt{3}(\varepsilon_1 - \varepsilon_2) & \varepsilon_1 + \varepsilon_2 + 4\varepsilon_3 & \sqrt{2}(\varepsilon_1 + \varepsilon_2 - 2\varepsilon_3) \\ \sqrt{6}(\varepsilon_1 - \varepsilon_2) & \sqrt{2}(\varepsilon_1 + \varepsilon_2 - 2\varepsilon_3) & 2(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \end{pmatrix}.\end{aligned}\tag{7.3.3}$$

When we define

$$\begin{aligned}\varepsilon_\pi &= \frac{1}{\sqrt{2}}(\varepsilon_1 - \varepsilon_2), \\ \varepsilon_\eta &= \frac{1}{\sqrt{6}}(\varepsilon_1 + \varepsilon_2 - 2\varepsilon_3), \\ \varepsilon_\sigma &= \frac{1}{\sqrt{3}}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3),\end{aligned}\tag{7.3.4}$$

we can rewrite (7.3.3) as

$$M_D = \frac{1}{\sqrt{3}} \begin{pmatrix} \varepsilon_\sigma + \frac{1}{\sqrt{2}}\varepsilon_\eta & \frac{1}{\sqrt{2}}\varepsilon_\pi & \varepsilon_\pi \\ \frac{1}{\sqrt{2}}\varepsilon_\pi & \varepsilon_\sigma - \frac{1}{\sqrt{2}}\varepsilon_\eta & \varepsilon_\eta \\ \varepsilon_\pi & \varepsilon_\eta & \sqrt{3}a + \varepsilon_\sigma \end{pmatrix}.\tag{7.3.5}$$

For the case  $a \gg \varepsilon_i$ , the matrix (7.3.5) can approximately be diagonalized by the rotation  $R_3(\theta_{12})$

$$R_3(\theta_{12}) = \begin{pmatrix} \cos \theta_{12} & \sin \theta_{12} & 0 \\ -\sin \theta_{12} & \cos \theta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix},\tag{7.3.6}$$

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<sup>5</sup>Y. Koide, Phys. Rev. **D39**, 1391 (1989)

where

$$\tan 2\theta_{12} = -\frac{\varepsilon_\pi}{\varepsilon_\eta}. \quad (7.3.7)$$

The approximate eigenvalues are given by

$$\begin{aligned} m_1 &\simeq \frac{1}{\sqrt{3}}\varepsilon_\sigma - \frac{1}{\sqrt{6}}\sqrt{\varepsilon_\pi^2 + \varepsilon_\eta^2}, \\ m_2 &\simeq \frac{1}{\sqrt{3}}\varepsilon_\sigma + \frac{1}{\sqrt{6}}\sqrt{\varepsilon_\pi^2 + \varepsilon_\eta^2}, \\ m_3 &\simeq a + \frac{1}{\sqrt{3}}\varepsilon_\sigma, \end{aligned} \quad (7.3.8)$$

where

$$\sqrt{\varepsilon_\pi^2 + \varepsilon_\eta^2} = \frac{2}{\sqrt{6}}\sqrt{\varepsilon_3^2 + \varepsilon_2^2 + \varepsilon_1^2 - (\varepsilon_3\varepsilon_2 + \varepsilon_2\varepsilon_1 + \varepsilon_1\varepsilon_3)}. \quad (7.3.9)$$

For  $\varepsilon_3^2 \gg \varepsilon_2^2 \gg \varepsilon_1^2$ , we obtain

$$\frac{m_1}{m_2} \simeq \frac{3\varepsilon_2}{4\varepsilon_3}. \quad (7.3.10)$$

## 8 Diagonalization of a nearly rank-1 matrix

Diagonalization of a nearly rank-1 matrix is discussed. It is an extended version of the democratic mass matrix. It is also useful to investigating a neutrino mass matrix.

### 8.1 Diagonalization of a rank-1 matrix

In general, a rank-1 matrix  $M$  has the following form<sup>6</sup>:

$$M = \begin{pmatrix} g_1^2 & g_1 g_2 & g_1 g_3 \\ g_1 g_2 & g_2^2 & g_2 g_3 \\ g_1 g_3 & g_2 g_3 & g_3^2 \end{pmatrix}. \quad (8.1.1)$$

The rank-1 matrix (8.1.1) is diagonalized by the following three ways:

(A)

$$R_{31}^T M R_{31} = D, \quad (8.1.2)$$

(B)

$$R_{12}^T M R_{12} = D, \quad (8.1.3)$$

(C)

$$R_{21}^T M R_{21} = D, \quad (8.1.4)$$

where

$$D = \text{diag}(0, 0, g_1^2 + g_2^2 + g_3^2), \quad (8.1.5)$$

$$R_{31} = R_3(\theta_{12})R_1(\theta_{23}), \quad (8.1.6)$$

$$R_{12} = R_1(\theta_{23})R_2(\theta_{13}), \quad (8.1.7)$$

$$R_{21} = R_2(\theta_{13})R_1(\theta_{23}), \quad (8.1.8)$$

$$R_1(\theta_{23}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}, \quad R_2(\theta_{13}) = \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix}, \quad R_3(\theta_{12}) = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (8.1.9)$$

---

<sup>6</sup>For convenience, we have taken the rank-1 matrix  $M$  real. If  $M$  is Hermitian, we can express the rank-1 matrix  $M$  as

$$M_{ij} = g_i g_j^*, \quad (8.1.1)'$$

instead of (8.1.1). Since we can obtain a real matrix  $\tilde{M}$  as

$$\tilde{M} = P^\dagger(\phi) M P(\phi),$$

where  $P = \text{diag}(e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3})$ , and  $g_i = |g_i| e^{i\phi_i}$ , because

$$\tilde{M}_{ij} = e^{-i\phi_i} g_i g_j^* e^{i\phi_j} = |g_i| |g_j|.$$

Therefore, we can always use the real matrix formulation which will be derived below (8.1.2).

i.e.

$$R_{31} = \begin{pmatrix} c_{12} & s_{12}c_{23} & s_{12}s_{23} \\ -s_{12} & c_{12}c_{23} & c_{12}s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}, \quad (8.1.10)$$

$$s_{12} = \frac{g_1}{\sqrt{g_1^2 + g_2^2}}, \quad c_{12} = \frac{g_2}{\sqrt{g_1^2 + g_2^2}}, \quad (8.1.11)$$

$$s_{23} = \frac{\sqrt{g_1^2 + g_2^2}}{\sqrt{g_1^2 + g_2^2 + g_3^2}}, \quad c_{23} = \frac{g_3}{\sqrt{g_1^2 + g_2^2 + g_3^2}}, \quad (8.1.12)$$

$$R_{12} = \begin{pmatrix} c_{13} & 0 & s_{13} \\ -s_{23}s_{13} & c_{23} & c_{13}s_{23} \\ -c_{23}s_{13} & -s_{23} & c_{13}c_{23} \end{pmatrix}, \quad (8.1.13)$$

$$s_{23} = \frac{g_2}{\sqrt{g_2^2 + g_3^2}}, \quad c_{23} = \frac{g_3}{\sqrt{g_2^2 + g_3^2}}, \quad (8.1.14)$$

$$s_{13} = \frac{g_1}{\sqrt{g_1^2 + g_2^2 + g_3^2}}, \quad c_{13} = \frac{\sqrt{g_2^2 + g_3^2}}{\sqrt{g_1^2 + g_2^2 + g_3^2}}, \quad (8.1.15)$$

$$R_{21} = \begin{pmatrix} c_{13} & -s_{13}s_{23} & s_{13}c_{23} \\ 0 & c_{23} & s_{23} \\ -s_{13} & -c_{13}s_{23} & c_{13}c_{23} \end{pmatrix}, \quad (8.1.16)$$

$$s_{13} = \frac{g_1}{\sqrt{g_1^2 + g_3^2}}, \quad c_{13} = \frac{g_3}{\sqrt{g_1^2 + g_3^2}}, \quad (8.1.18)$$

$$s_{23} = \frac{g_2}{\sqrt{g_1^2 + g_2^2 + g_3^2}}, \quad c_{23} = \frac{\sqrt{g_1^2 + g_3^2}}{\sqrt{g_1^2 + g_2^2 + g_3^2}}. \quad (8.1.19)$$

The reason that we have three ways to the diagonalization is due to the fact that two of the eigenvalues,  $m_1$  and  $m_2$  are degenerated as  $m_1 = m_2 = 0$ , so that we have a degree of freedom of a further rotation  $R_3(\theta_{12})$ . For example, the general mixing matrix  $U$  is given by

$$U = R_1(\theta_{23})R_2(\theta_{13})R_3(\theta_{12}), \quad (8.1.20)$$

where  $\theta_{23}$  and  $\theta_{13}$  are given by (8.1.14) and (8.1.15), respectively, while  $\theta_{12}$  is free. The value of  $\theta_{12}$  will be fixed by an additional term to the rank-1 matrix as we show in the next subsection.

## 8.2 Diagonalization of a nearly rank-1 matrix

We denote a nearly rank-1 matrix as

$$M_{ij} = g_i g_j + \varepsilon_{ij}, \quad (8.2.1)$$

where  $\varepsilon$  is a small term with  $\varepsilon^T = \varepsilon$ . As we stated in the previous subsection, there are three ways to the diagonalization of the rank-1 matrix. According to the ways (8.1.2)-(8.1.4), we will show three ways to the diagonalization of the matrix (8.2.1).

### (A) Diagonalization by the rotation $R_{31}$

When we transform the matrix  $M$  by the rotation  $R_{31}$ , (8.1.10), as

$$M' = R_{31}^T M R_{31}, \quad (8.2.2)$$

we obtain

$$M'_{11} = \frac{1}{g_1^2 + g_2^2} (\varepsilon_{11} g_2^2 - 2\varepsilon_{12} g_2 g_1 + \varepsilon_{22} g_1^2), \quad (8.2.3)$$

$$M'_{22} = \frac{1}{(g_1^2 + g_2^2)(g_1^2 + g_2^2 + g_3^2)} \left[ (\varepsilon_{22} g_2^2 + 2\varepsilon_{12} g_1 g_2 + \varepsilon_{11} g_1^2) g_3^2 - 2(\varepsilon_{13} g_1 + \varepsilon_{23} g_2)(g_1^2 + g_2^2) \right], \quad (8.2.4)$$

$$M'_{33} = g_1^2 + g_2^2 + g_3^2 + \frac{1}{g_1^2 + g_2^2 + g_3^2} \left[ \varepsilon_{33} g_3^2 + 2(\varepsilon_{23} g_2 + \varepsilon_{13} g_1) g_3 + 2\varepsilon_{12} g_1 g_2 + \varepsilon_{22} g_2^2 + \varepsilon_{11} g_1^2 \right], \quad (8.2.5)$$

$$M'_{12} = \frac{1}{(g_1^2 + g_2^2)\sqrt{g_1^2 + g_2^2 + g_3^2}} \left\{ \left[ \varepsilon_{12}(g_2^2 - g_1^2) - (\varepsilon_{22} - \varepsilon_{11})g_1 g_2 \right] g_3 - (\varepsilon_{13} g_2 - \varepsilon_{23} g_1)(g_1^2 + g_2^2) \right\}, \quad (8.2.6)$$

$$M'_{13} = \frac{1}{\sqrt{g_1^2 + g_2^2}\sqrt{g_1^2 + g_2^2 + g_3^2}} \left[ (\varepsilon_{13} g_2 - \varepsilon_{23} g_1) g_3 + \varepsilon_{12}(g_2^2 - g_1^2) - (\varepsilon_{22} - \varepsilon_{11})g_1 g_2 \right], \quad (8.2.7)$$

$$M'_{23} = \frac{1}{\sqrt{g_1^2 + g_2^2}(g_1^2 + g_2^2 + g_3^2)} \left[ (\varepsilon_{13} g_1 + \varepsilon_{23} g_2)(g_3^2 - g_2^2 - g_1^2) + (\varepsilon_{22} - \varepsilon_{33})g_2^2 g_3 + 2\varepsilon_{12} g_1 g_2 g_3 + (\varepsilon_{11} - \varepsilon_{33})g_1^2 g_3 \right]. \quad (8.2.8)$$

For a case  $|M_{33}| \gg |M_{ij}|$  ( $i \neq 3, j \neq 3$ ) ( $g_3^2 \gg g_2^2 + g_1^2$ ), the matrix  $M'$  is approximately diagonalized by an additional rotation  $R_3(\theta'_{12})$ :

$$R_3^T(\theta'_{12}) R_{31}^T M R_{31} R_3(\theta'_{12}) \simeq D. \quad (8.2.9)$$

The rotation  $R_{31} R_3(\theta'_{12})$  is explicitly expressed as

$$R_{31} R_3(\theta'_{12}) = \begin{pmatrix} c_{12} c'_{12} - s_{12} c_{23} s'_{12} & c_{12} s'_{12} + s_{12} c_{23} c'_{12} & s_{12} s_{23} \\ -s_{12} c'_{12} - c_{12} c_{23} s'_{12} & -s_{12} s'_{12} + c_{12} c_{23} c'_{12} & c_{12} s_{23} \\ s_{23} s'_{12} & -s_{23} c'_{12} & c_{23} \end{pmatrix}. \quad (8.2.10)$$

Since

$$M'_{12} \simeq \frac{1}{g_1^2 + g_2^2} \left[ \varepsilon_{12}(g_2^2 - g_1^2) - (\varepsilon_{22} - \varepsilon_{11})g_1g_2 \right], \quad (8.2.11)$$

$$M'_{22} - M'_{11} \simeq \frac{1}{g_1^2 + g_2^2} \left[ (\varepsilon_{22} - \varepsilon_{11})(g_2^2 - g_1^2) + 4\varepsilon_{12}g_1g_2 \right], \quad (8.2.12)$$

we obtain the value of  $\theta'_{12}$  as

$$\tan 2\theta'_{12} \simeq \frac{2 \left[ \varepsilon_{12}(g_2^2 - g_1^2) - (\varepsilon_{22} - \varepsilon_{11})g_1g_2 \right]}{(\varepsilon_{22} - \varepsilon_{11})(g_2^2 - g_1^2) + 4\varepsilon_{12}g_1g_2}. \quad (8.2.13)$$

When we define

$$\tan 2\theta_\varepsilon = \frac{2\varepsilon_{12}}{\varepsilon_{22} - \varepsilon_{11}}, \quad (8.2.14)$$

$$\tan 2\theta_g = \frac{2g_1g_2}{g_2^2 - g_1^2}, \quad (8.2.15)$$

$$\left( \Rightarrow \tan \theta_g = \frac{g_1}{g_2} \right), \quad (8.2.16)$$

we find

$$\theta'_{12} \simeq \theta_\varepsilon - \theta_g, \quad (8.2.17)$$

from Eq.(8.2.13) .

Note that, for the case with  $\varepsilon_{33} \gg \varepsilon_{22} \gg \varepsilon_{11}$ , the approximation (8.2.12) [also (8.2.13)] is not valid. For example, for the case

$$\varepsilon_{ij} = \delta_{ij}\varepsilon g_i^2, \quad (8.18)$$

we obtain

$$\tan 2\theta'_{12} = -\frac{g_1g_2g_3(g_2^2 - g_1^2)\sqrt{g_1^2 + g_2^2 + g_3^2}}{(g_1^4 + g_2^4)g_3^2 - (g_1^2 + g_2^2)g_1^2g_2^2}. \quad (8.2.19)$$

### (B) Diagonalization by the rotation $R_{12}$

Similarly, when we transform  $M$  as

$$M' = R_{12}^T M R_{12}, \quad (8.2.20)$$

we obtain

$$\begin{aligned} M'_{11} = & \frac{1}{(g_2^2 + g_3^2)(g_1^2 + g_2^2 + g_3^2)} \left[ \varepsilon_{11}g_3^4 \right. \\ & - 2\varepsilon_{13}g_1g_3^3 + 2(\varepsilon_{11}g_2^2 + \varepsilon_{33}g_1^2 - \varepsilon_{12}g_1g_2)g_3^2 \\ & \left. - 2(\varepsilon_{13}g_2 - \varepsilon_{23}g_1)g_1g_2g_3 + (\varepsilon_{11}g_2^2 - 2\varepsilon_{12}g_1g_2 + \varepsilon_{22}g_1^2)g_2^2 \right], \end{aligned} \quad (8.2.21)$$

$$M'_{22} = \frac{1}{g_2^2 + g_3^2} \left[ \varepsilon_{22}g_3^2 - 2\varepsilon_{23}g_2g_3 + \varepsilon_{33}g_2^2 \right], \quad (8.2.22)$$

$$\begin{aligned} M'_{33} = & g_1^2 + g_2^2 + g_3^2 + \frac{1}{g_1^2 + g_2^2 + g_3^2} \left[ \varepsilon_{33}g_3^2 \right. \\ & \left. + 2(\varepsilon_{13}g_1 + \varepsilon_{23}g_2)g_3 + \varepsilon_{22}g_2^2 + 2\varepsilon_{12}g_1g_2 + \varepsilon_{11}g_1^2 \right], \end{aligned} \quad (8.2.23)$$

$$\begin{aligned} M'_{12} = & \frac{1}{(g_2^2 + g_3^2)\sqrt{g_1^2 + g_2^2 + g_3^2}} \left[ \varepsilon_{12}g_3^3 - (\varepsilon_{13}g_2 + \varepsilon_{23}g_1)g_3^2 \right. \\ & \left. + (\varepsilon_{33} - \varepsilon_{22})g_1g_2g_3 + (\varepsilon_{12}g_3 - \varepsilon_{13}g_2 + \varepsilon_{23}g_1)g_3^2 \right], \end{aligned} \quad (8.2.24)$$

$$M'_{13} = \frac{1}{\sqrt{g_2^2 + g_3^2}(g_1^2 + g_2^2 + g_3^2)} [\varepsilon_{13}g_3^3 + (\varepsilon_{12}g_2 + (\varepsilon_{11} - \varepsilon_{33})g_1)g_3^2 + (\varepsilon_{13}g_2^2 - 2\varepsilon_{23}g_1g_2 - \varepsilon_{13}g_1^2)g_3 + \varepsilon_{12}g_2^3 - (\varepsilon_{22} - \varepsilon_{11})g_1g_2^2 - \varepsilon_{12}g_1^2g_2], \quad (8.2.25)$$

$$M'_{23} = \frac{1}{\sqrt{g_2^2 + g_3^2}\sqrt{g_1^2 + g_2^2 + g_3^2}} [\varepsilon_{23}g_3^2 - ((\varepsilon_{33} - \varepsilon_{22})g_2 - \varepsilon_{12}g_1)g_3 - \varepsilon_{23}g_2^2 - \varepsilon_{13}g_1g_2]. \quad (8.2.26)$$

For a case with  $g_3^2 \gg g_2^2 + g_1^2$ , the matrix  $M'$  is approximately diagonalized by an additional rotation  $R_3(\theta'_{12})$  as

$$R_3^T(\theta'_{12})R_{12}^TMR_{12}R_3(\theta'_{12}) \simeq D. \quad (8.2.27)$$

From the relations

$$\begin{aligned} M'_{12} &\simeq \varepsilon_{12} \\ M'_{22} - M'_{11} &\simeq \varepsilon_{22} - \varepsilon_{11}, \end{aligned} \quad (8.2.28)$$

we find that the rotation angle  $\theta'_{12}$  is given by

$$\tan 2\theta'_{12} \simeq \frac{2\varepsilon_{12}}{\varepsilon_{22} - \varepsilon_{11}}. \quad (8.2.29)$$

### (C) Diagonalization by the rotation $R_{21}$

Similarly, the matrix elements of

$$M' = R_{21}^TMR_{21}, \quad (8.2.30)$$

are given as follows:

$$M'_{11} = \frac{1}{g_1^2 + g_3^2} [\varepsilon_{11}g_3^2 - 2\varepsilon_{13}g_1g_3 + \varepsilon_{33}g_1^2], \quad (8.2.31)$$

$$\begin{aligned} M'_{22} &= \frac{1}{(g_1^2 + g_3^2)(g_1^2 + g_2^2 + g_3^2)} [\varepsilon_{22}g_3^4 - 2\varepsilon_{23}g_2g_3^3 \\ &+ (2\varepsilon_{22}g_1^2 - 2\varepsilon_{12}g_1g_2 + \varepsilon_{33}g_2^2)g_3^2 + 2(\varepsilon_{13}g_1g_2^2 - \varepsilon_{23}g_1^2g_2)g_3 \\ &+ \varepsilon_{22}g_1^4 + \varepsilon_{11}g_1^2g_2^2 - 2\varepsilon_{12}g_1^3g_2], \end{aligned} \quad (8.2.32)$$

$$\begin{aligned} M'_{33} &= g_1^2 + g_2^2 + g_3^2 + \frac{1}{g_1^2 + g_2^2 + g_3^2} [\varepsilon_{33}g_3^2 \\ &+ 2(\varepsilon_{13}g_1 + \varepsilon_{23}g_2)g_3 + \varepsilon_{22}g_2^2 + 2\varepsilon_{12}g_1g_2 + \varepsilon_{11}g_1^2], \end{aligned} \quad (8.2.33)$$

$$\begin{aligned} M'_{12} &= \frac{1}{(g_1^2 + g_3^2)\sqrt{g_1^2 + g_2^2 + g_3^2}} [\varepsilon_{12}g_3^3 \\ &- (\varepsilon_{13}g_2 + \varepsilon_{23}g_1)g_3^2 + (\varepsilon_{33} - \varepsilon_{11})g_1g_2g_3 + \varepsilon_{12}g_1^2g_3 + (\varepsilon_{13}g_2 - \varepsilon_{23}g_1)g_1^2], \end{aligned} \quad (8.2.34)$$

$$\begin{aligned} M'_{13} &= \frac{1}{\sqrt{g_1^2 + g_3^2}\sqrt{g_1^2 + g_2^2 + g_3^2}} \{ \varepsilon_{13}g_3^2 \\ &+ [\varepsilon_{12}g_2 - (\varepsilon_{33} - \varepsilon_{11})g_1]g_3 - (\varepsilon_{23}g_2 + \varepsilon_{13}g_1)g_1 \}, \end{aligned} \quad (8.2.35)$$

$$\begin{aligned} M'_{23} &= \frac{1}{\sqrt{g_1^2 + g_3^2}(g_1^2 + g_2^2 + g_3^2)} \{ \varepsilon_{23}g_3^2 \\ &- [(\varepsilon_{33} - \varepsilon_{22})g_2 + \varepsilon_{12}g_1]g_3^2 - (\varepsilon_{23}g_2^2 + 2\varepsilon_{13}g_1g_2 - \varepsilon_{23}g_1^2)g_3 \} \end{aligned}$$

$$-\varepsilon_{12}g_1g_2^2 + (\varepsilon_{22} - \varepsilon_{11})g_1^2 + \varepsilon_{12}g_1^3\}. \quad (8.2.36)$$

For a case with  $g_3^2 \gg g_2^2 + g_1^2$ , the matrix  $M'$  is approximately diagonalized by an additional rotation  $R_3(\theta'_{12})$  as

$$R_3^T(\theta'_{12})R_{21}^TMR_{21}R_3(\theta'_{12}) \simeq D, \quad (8.2.37)$$

and we also find

$$\tan 2\theta'_{12} \simeq \frac{2\varepsilon_{12}}{\varepsilon_{22} - \varepsilon_{11}}. \quad (8.2.38)$$

## 9 Diagonalization of Mass Matrices with Specific Forms

The eigenvalues and mixing matrix of a mass matrix with a specific form are discussed. Some of them are motivated from the neutrino phenomenology.

### 9.1 NNI-type mass matrix

The following mass matrix is called the nearest-neighbor interaction (NNI) type mass matrix:

$$\widehat{M} = P^\dagger(\delta_L)\overline{M}P(\delta_R), \quad (9.1.1)$$

$$\overline{M} = \begin{pmatrix} 0 & c_1 & 0 \\ c_2 & 0 & b_1 \\ 0 & b_2 & a \end{pmatrix}, \quad (9.1.2)$$

where  $a$ ,  $b_i$  and  $c_i$  are real parameters, and  $P(\delta)$  is a phase matrix

$$P(\delta) = \text{diag}(e^{i\delta_1}, e^{i\delta_2}, e^{i\delta_3}). \quad (9.1.3)$$

The matrix (9.1.1) with  $c_1 = c_2$  and  $b_1 = b_2$  is known as the Fritzsch mass matrix.

#### 9.1.1 Branco-Lavoura-Mota theorem

The following theorem has been given by Branco, Lavoura and Mota <sup>7</sup>.

**[Theorem]**

Any quark mass matrices  $(M_u, M_d)$  can always be transformed into NNI-type mass matrices  $(\widehat{M}_u, \widehat{M}_d)$  by re-basing without losing generality.

Here, the ‘‘re-basing’’ means the re-definition of the flavor basis by the transformation

$$\begin{aligned} M_u &\rightarrow M'_u = A^\dagger M_u B_u, \\ M_d &\rightarrow M'_d = A^\dagger M_d B_d, \end{aligned} \quad (9.1.4)$$

where  $A$ ,  $B_u$  and  $B_d$  are  $3 \times 3$  unitary matrices. Note that this statement is based on the  $SU(2)_L \times U(1)_Y$  gauge model. If we adopt a  $SU(2)_L \times SU(2)_R$  model, the re-basing (9.1.4) must be replaced by

$$\begin{aligned} M_u &\rightarrow A^\dagger M_u B, \\ M_d &\rightarrow A^\dagger M_d B. \end{aligned} \quad (9.1.5)$$

We know that we can always transform arbitrary quark mass matrices  $(M_u, M_d)$  into Hermitian mass matrices. However, we cannot transform  $(M_u, M_d)$  into  $(\widehat{M}_u, \widehat{M}_d)$  with the NNI form, and besides, with  $\widehat{M}_u^\dagger = \widehat{M}_u$  and  $\widehat{M}_d^\dagger = \widehat{M}_d$ .

#### 9.1.2 Description with 10 observable parameters

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<sup>7</sup>G. C. Branco, L. Lavoura and F. Mota, Phys. Rev.**D33**, 860 (1986).

Generally,  $n \times n$  quark mass matrices ( $M_u, M_d$ ) have many parameters (maximally  $2n^2$ ). On the other hand, we have 10 observable quantities (3 + 3 masses and 4 Cabibbo-Kobayashi-Maskawa (CKM) matrix parameters<sup>8</sup>) for  $n = 3$ . Let us show that if we choose a suitable NNI form, we can describe the quark mass matrices ( $M_u, M_d$ ) with only 7 parameters.

First, we define an Hermitian matrix  $\widehat{H}_q$  by

$$\widehat{H}_q = \widehat{M}_q \widehat{M}_q^\dagger = P^\dagger(\delta_{qL}) \overline{M}_q \overline{M}_q^T P(\delta_{qL}) \equiv P^\dagger(\delta_{qL}) \overline{H}_q P(\delta_{qL}), \quad (9.1.6)$$

where  $q = u, d$  and

$$\overline{H}_q = \begin{pmatrix} c_{q1}^2 & 0 & b_{q2} c_{q1} \\ 0 & b_{q1}^2 + c_{q2}^2 & a_q b_{q1} \\ b_{q2} c_{q1} & a_q b_{q1} & a_q^2 + b_{q2}^2 \end{pmatrix}. \quad (9.1.7)$$

(Hereafter, we drop the index  $q$ .) We denote the number of the independent parameters in the matrix  $M$  as  $N(M)$ . Since  $N(\overline{M}) = 5$  and  $N(\overline{H}) = 5$ , we can completely determine the parameters  $a, b_1, b_2, c_1$  and  $c_2$  from the values  $\overline{H}_{ij}$ :

$$\begin{aligned} a &= \frac{1}{\sqrt{\overline{H}_{11}}} \sqrt{\overline{H}_{11} \overline{H}_{33} - \overline{H}_{31}^2}, \\ b_1 &= \frac{\overline{H}_{23} \sqrt{\overline{H}_{11}}}{\sqrt{\overline{H}_{11} \overline{H}_{33} - \overline{H}_{31}^2}}, \quad b_2 = \frac{\overline{H}_{13}}{\sqrt{\overline{H}_{11}}}, \\ c_1 &= \sqrt{\overline{H}_{11}}, \quad c_2 = \frac{\sqrt{\det \overline{H}}}{\sqrt{\overline{H}_{11} \overline{H}_{33} - \overline{H}_{31}^2}}. \end{aligned} \quad (9.1.8)$$

On the other hand, the Hermitian matrices ( $H_u, H_d$ ) are always rewritten by the form

$$\begin{aligned} H_u &= A^\dagger D_u^2 A, \\ H_d &= A^\dagger V D_d^2 V^\dagger A, \end{aligned} \quad (9.1.9)$$

where  $V$  is the CKM matrix

$$V = U_{uL}^\dagger U_{dL}, \quad (9.1.10)$$

and

$$\begin{aligned} D_u &= \text{diag}(m_u, m_c, m_t), \\ D_d &= \text{diag}(m_d, m_s, m_b). \end{aligned} \quad (9.1.11)$$

When we choose an orthogonal matrix  $A$ , the number of the parameters is  $N(A) = 3$ . Since we have 3 constraints, i.e.,

$$(H_u)_{12} = 0, \quad \text{Re}(H_d)_{12} = 0, \quad \text{Im}(H_d)_{12} = 0, \quad (9.1.12)$$

[note that  $(H_u)_{ij}$  are real], we can explicitly determine all of the parameters of  $A$  from the observable quantities  $D_u, D_d$  and  $V$ . Thus, we can write all of the parameters of the matrices ( $\widehat{H}_u, \widehat{H}_d$ ) in terms of the observable quantities, so that we can also write all of the parameters of the matrices ( $\widehat{M}_u, \widehat{M}_d$ ) in terms of the observable quantities by using (9.1.8).

## 9.2 Mass matrix with a specific zero-texture

### 9.2.1 Mass matrix with $M_{11} = M_{22} = M_{12} = M_{21} = 0$

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<sup>8</sup>see Chap.12

We diagonalize the following mass matrix with four texture zeros

$$M = \begin{pmatrix} 0 & 0 & a_1 \\ 0 & 0 & a_2 \\ a_1 & a_2 & 2a_3 \end{pmatrix}. \quad (9.2.1)$$

The eigenvalues are given by

$$\begin{aligned} m_1 &= 0, \\ -m_2 &= a_3 - \sqrt{a_1^2 + a_2^2 + a_3^2}, \\ m_3 &= a_3 + \sqrt{a_1^2 + a_2^2 + a_3^2}. \end{aligned} \quad (9.2.2)$$

The mixing matrix  $U$  is given by

$$U^\dagger M U = \text{diag}(m_1, -m_2, m_3), \quad (9.2.3)$$

$$U = \begin{pmatrix} c_{12} & s_{12}c_{23} & s_{12}s_{23} \\ -s_{12} & c_{12}c_{23} & c_{12}s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}, \quad (9.2.4)$$

where

$$s_{12} = \frac{a_1}{\sqrt{a_1^2 + a_2^2}}, \quad c_{12} = \frac{a_2}{\sqrt{a_1^2 + a_2^2}}, \quad (9.2.5)$$

$$s_{23} = \frac{\sqrt{\sqrt{a_1^2 + a_2^2 + a_3^2} - a_3}}{\sqrt{2}(a_1^2 + a_2^2 + a_3^2)^{\frac{1}{4}}}, \quad c_{23} = \frac{\sqrt{\sqrt{a_1^2 + a_2^2 + a_3^2} + a_3}}{\sqrt{2}(a_1^2 + a_2^2 + a_3^2)^{\frac{1}{4}}}. \quad (9.2.6)$$

From Eq.(9.2.2), we can rewrite the relations (9.2.6) as follows;

$$s_{23} = \sqrt{\frac{m_2}{m_3 + m_2}}, \quad c_{23} = \sqrt{\frac{m_3}{m_3 + m_2}}. \quad (9.2.7)$$

### 9.2.2 $4 \times 4$ mass matrix with $M_{ij} = \delta_{4i}x_i + \delta_{4j}x_j$

We can also give eigenvalues and mixing matrix of the following  $4 \times 4$  matrix

$$M = \begin{pmatrix} 0 & 0 & 0 & x_1 \\ 0 & 0 & 0 & x_2 \\ 0 & 0 & 0 & x_3 \\ x_1 & x_2 & x_3 & x_0 \end{pmatrix}. \quad (9.2.8)$$

The mixing matrix  $R$  is given by

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ 0 & r_{32} & r_{33} & r_{34} \\ 0 & 0 & r_{43} & r_{44} \end{pmatrix}, \quad (9.2.9)$$

where

$$r_{11} = \frac{x_2}{N_1}, \quad r_{21} = -\frac{x_1}{N_1}, \quad (9.2.10)$$

$$r_{12} = x_1^2 x_3 \frac{1}{N_2}, \quad r_{22} = x_1 x_2 x_3 \frac{1}{N_2}, \quad r_{32} = -x_1(x_2^2 + x_1^2) \frac{1}{N_2}, \quad (9.2.11)$$

$$r_{13} = 2x_1 \frac{1}{N_3}, \quad r_{23} = 2x_2 \frac{1}{N_3}, \quad r_{33} = 2x_3 \frac{1}{N_3}, \quad r_{43} = -(\sqrt{A} - x_0) \frac{1}{N_3}, \quad (9.2.12)$$

$$r_{14} = 2x_1 \frac{1}{N_4}, \quad r_{24} = 2x_2 \frac{1}{N_4}, \quad r_{34} = 2x_3 \frac{1}{N_4}, \quad r_{44} = (\sqrt{A} + x_0) \frac{1}{N_4}, \quad (9.2.13)$$

$$N_1 = \sqrt{x_2^2 + x_1^2}, \quad (9.2.14)$$

$$N_2 = x_1 \sqrt{x_2^2 + x_1^2} \sqrt{x_3^2 + x_2^2 + x_1^2}, \quad (9.2.15)$$

$$N_3 = \sqrt{2\sqrt{A}(\sqrt{A} - x_0)}, \quad (9.2.16)$$

$$N_4 = \sqrt{2\sqrt{A}(\sqrt{A} + x_0)}. \quad (9.2.17)$$

$$\sqrt{A} = \sqrt{4(x_3^2 + x_2^2 + x_1^2) + x_0^2}. \quad (9.2.18)$$

The eigenvalues are given by

$$R^T M R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{pmatrix}, \quad (9.2.19)$$

$$\begin{aligned} m_3 &= -\frac{1}{2}(\sqrt{A} - x_0), \\ m_4 &= \frac{1}{2}(\sqrt{A} + x_0). \end{aligned} \quad (9.2.20)$$

### 9.3 Zee mass matrix

#### 9.3.1 Mass eigenvalues

The mass matrix

$$M = m_0 \begin{pmatrix} 0 & z_3 & z_2 \\ z_3 & 0 & z_1 \\ z_2 & z_1 & 0 \end{pmatrix}, \quad (9.3.1)$$

is called the Zee mass matrix, which was first proposed by Zee<sup>9</sup> from a radiatively induced mass generation mechanism.

For the matrix (9.3.1), we can obtain the following relations :

$$\text{Tr}M = m_1 + m_2 + m_3 = 0, \quad (9.3.2)$$

$$\text{Tr}M^2 = m_1^2 + m_2^2 + m_3^2 = 2(z_1^2 + z_2^2 + z_3^2)m_0^2, \quad (9.3.3)$$

$$\det M = m_1 m_2 m_3 = 2z_1 z_2 z_3 m_0^3. \quad (9.3.4)$$

From (9.3.2) and (9.3.3), we obtain

$$m_1 m_2 + m_2 m_3 + m_3 m_1 = -(z_1^2 + z_2^2 + z_3^2)m_0^2, \quad (9.3.5)$$

<sup>9</sup>A. Zee, Phys. Lett. **93B**, 389 (1980); L. Wolfenstein, Nucl. Phys. **B175**, 93 (1980); S. T. Petcov, Phys. Lett. **115B**, 401 (1982).

Therefore, the eigenvalues  $m_i$  ( $i = 1, 2, 3$ ) are given as solutions of the equation

$$m_i^3 - (z_1^2 + z_2^2 + z_3^2)m_0^2 m_i - 2z_1 z_2 z_3 m_0^3 = 0. \quad (9.3.6)$$

By adjusting the normalization parameter  $m_0$ , without losing the generality, we can put

$$z_1^2 + z_2^2 + z_3^2 = 1. \quad (9.3.7)$$

Then, the solutions  $m_i \equiv m_0 \lambda_i$  are given by

$$\lambda_i^3 - \lambda_i - 2q = 0, \quad (9.3.8)$$

where

$$q = z_1 z_2 z_3. \quad (9.3.9)$$

The mass levels  $m_i$  versus  $q$  are illustrated in Fig. 9.3.1 and 9.3.2.

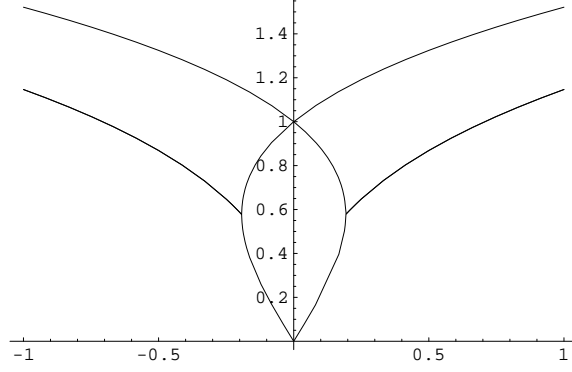


Fig. 9.3.1: Behaviors of  $|m_i|$  versus the parameter  $q = z_1 z_2 z_3$ .

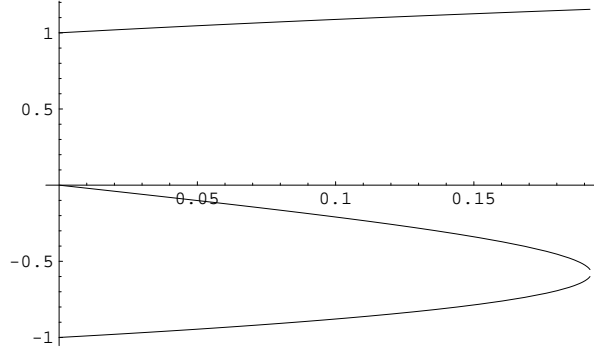


Fig. 9.3.2: Behaviors of  $m_i$  versus the parameter  $q = z_1 z_2 z_3$  in the range  $0 \leq q \leq 1/\sqrt{27}$ .

For a small value of  $q$ , Eq.(9.3.8) is written as

$$\lambda_i(\lambda_i^2 - 1) = 2q \equiv 2\varepsilon \simeq 0, \quad (9.3.10)$$

so that we can consider the following solutions

$$\lambda_1 = 1 + \varepsilon_1, \quad \lambda_2 = -1 + \varepsilon_2, \quad \lambda_3 = \varepsilon_3. \quad (9.3.11)$$

By putting (9.3.11) into (9.3.10), we obtain

$$(1 + \varepsilon_1)(1 + 2\varepsilon_1 + \varepsilon_1^2 - 1) = 2\varepsilon \Rightarrow \varepsilon_1 \simeq \varepsilon(1 - \frac{3}{2}\varepsilon), \quad (9.3.12)$$

$$(-1 + \varepsilon_2)(1 - 2\varepsilon_2 + \varepsilon_2^2 - 1) = 2\varepsilon \Rightarrow \varepsilon_2 \simeq \varepsilon(1 + \frac{3}{2}\varepsilon), \quad (9.3.13)$$

$$\varepsilon_3(\varepsilon_3^2 - 1) = 2\varepsilon \Rightarrow \varepsilon_3 \simeq -2\varepsilon(1 + 4\varepsilon^2), \quad (9.3.14)$$

i.e.,

$$\lambda_1 \simeq 1 + \varepsilon - \frac{3}{2}\varepsilon^2, \quad (9.3.15)$$

$$\lambda_2 \simeq -(1 - \varepsilon - \frac{3}{2}\varepsilon^2), \quad (9.3.16)$$

$$\lambda_3 \simeq -2\varepsilon(1 + 4\varepsilon^2), \quad (9.3.17)$$

where  $\varepsilon = q = z_1 z_2 z_3$ . From (9.3.15)-(9.3.17),

$$\Delta m_{12}^2 = m_1^2 - m_2^2 \simeq [(1 + \varepsilon)^2 - (1 - \varepsilon)^2]m_0^2 \simeq 4\varepsilon m_0^2, \quad (9.3.18)$$

$$\Delta m_{23}^2 = m_2^2 - m_3^2 \simeq [(1 - \varepsilon)^2 - 4\varepsilon^2]m_0^2 \simeq (1 - 2\varepsilon)m_0^2. \quad (9.3.19)$$

If we regard  $\Delta m_{12}^2$  and  $\Delta m_{23}^2$  as

$$\Delta m_{solar}^2 = \Delta m_{12}^2 \simeq 4\varepsilon m_0^2, \quad (9.3.20)$$

$$\Delta m_{atm}^2 = \Delta m_{23}^2 \simeq m_0^2, \quad (9.3.21)$$

respectively, we can estimate the value of  $\varepsilon$  as follows:

$$\varepsilon \simeq \frac{1}{4} \frac{\Delta m_{solar}^2}{\Delta m_{atm}^2} \simeq \frac{1}{4} \cdot \frac{7.9 \times 10^{-5} \text{eV}^2}{2.74 \times 10^{-3} \text{eV}^2} = 7.2 \times 10^{-3}. \quad (9.3.22)$$

(We also may regard  $\Delta m_{12}^2 = -\Delta m_{solar}^2$ . Then, we can obtain  $\varepsilon \simeq -7.2 \times 10^{-3}$ .)

### 9.3.2 Mixing Matrix

On the other hand, the mixing matrix  $U$  is obtained as follows:

$$U^T M U = D = \text{diag}(m_1, m_2, m_3) = m_0 \text{diag}(\lambda_1, \lambda_2, \lambda_3), \quad (9.3.23)$$

i.e.,

$$\sum_k M_{ik} U_{kj} = U_{ij} D_{jj}, \quad (9.3.24)$$

where  $M$  is given by Eq.(9.3.1). The relations (9.3.24) can explicitly be written as follows:

$$-\lambda_j U_{1j} + z_3 U_{2j} + z_2 U_{3j} = 0, \quad (9.3.25)$$

$$z_3 U_{1j} - \lambda_j U_{2j} + z_1 U_{3j} = 0, \quad (9.3.26)$$

$$z_2 U_{1j} + z_1 U_{2j} - \lambda_j U_{3j} = 0. \quad (9.3.27)$$

By eliminating  $U_{3j}$  from (9.3.25) and (9.3.26), we obtain

$$\frac{U_{1j}}{U_{2j}} = \frac{z_2 \lambda_j + z_1 z_3}{z_1 \lambda_j + z_2 z_3}, \quad (9.3.28)$$

and also by eliminating  $U_{1j}$ , we obtain

$$\frac{U_{3j}}{U_{2j}} = \frac{\lambda_j^2 - z_3^2}{z_1\lambda_j + z_2z_3}. \quad (9.3.29)$$

Therefore, we obtain

$$U_{1j} = \frac{1}{N_j}(z_2\lambda_j + z_1z_3), \quad (9.3.30)$$

$$U_{2j} = \frac{1}{N_j}(z_1\lambda_j + z_1z_3), \quad (9.3.31)$$

$$U_{3j} = \frac{1}{N_j}(\lambda_j^2 - z_3^2), \quad (9.3.32)$$

where

$$\begin{aligned} N_j^2 &= (z_2\lambda_j + z_1z_3)^2 + (z_1\lambda_j + z_1z_3)^2 + (\lambda_j^2 - z_3^2)^2 \\ &= \lambda_j^4 + (z_2^2 + z_1^2 - 2z_3^2)\lambda_j^2 + 4z_1z_2z_3\lambda_j + z_3^2(z_3^2 + z_2^2 + z_1^2) \\ &= \lambda_j^4 + (1 - 3z_3^2)\lambda_j^2 + 4\epsilon\lambda_j + z_3^2. \end{aligned} \quad (9.3.33)$$

By using Eq.(9.3.10), we can write (9.3.33) as

$$N_j^2 = (3\lambda_j^2 - 1)(\lambda_j^2 - z_3^2). \quad (9.3.34)$$

For the eigenvalues (9.3.15)-(9.3.17), we obtain

$$N_1^2 \simeq [3(1 + \epsilon)^2 - 1][(1 + \epsilon)^2 - z_3^2] \simeq 2(1 - z_3^2) \left(1 + 3\epsilon + \frac{2\epsilon}{1 - z_3^2}\right), \quad (9.3.35)$$

$$U_{11} \simeq \frac{1}{N_1}[z_2(1 + \epsilon) + z_1z_3] \simeq \frac{1}{\sqrt{2}} \frac{z_2}{\sqrt{1 - z_3^2}} \left(1 - \frac{1}{2}\epsilon + \frac{z_3}{z_2}z_1 - \frac{\epsilon}{1 - z_3^2}\right), \quad (9.3.36)$$

$$U_{21} \simeq \frac{1}{N_1}[z_1(1 + \epsilon) + z_2z_3] \simeq \frac{1}{\sqrt{2}} \frac{z_2z_3}{\sqrt{1 - z_3^2}} \left(1 + \frac{z_1}{z_2z_3} - \frac{3}{2}\epsilon - \frac{\epsilon}{1 - z_3^2}\right), \quad (9.3.37)$$

$$U_{31} \simeq \frac{1}{N_1}[(1 + \epsilon)^2 - z_3^2] \simeq \frac{\sqrt{1 - z_3^2}}{\sqrt{2}} \left(1 - \frac{3}{2}\epsilon + \frac{\epsilon}{1 - z_3^2}\right), \quad (9.3.38)$$

$$N_2^2 \simeq [3(1 - \epsilon)^2 - 1][(1 - \epsilon)^2 - z_3^2] \simeq 2(1 - z_3^2) \left(1 - 3\epsilon - \frac{2\epsilon}{1 - z_3^2}\right), \quad (9.3.39)$$

$$U_{12} \simeq \frac{1}{N_2}[-z_2(1 - \epsilon) + z_1z_3] \simeq -\frac{1}{\sqrt{2}} \frac{z_2}{\sqrt{1 - z_3^2}} \left(1 - \frac{1}{2}\epsilon - \frac{z_3}{z_2}z_1 + \frac{\epsilon}{1 - z_3^2}\right), \quad (9.3.40)$$

$$U_{22} \simeq \frac{1}{N_2}[-z_1(1 - \epsilon) + z_2z_3] \simeq \frac{1}{\sqrt{2}} \frac{z_2z_3}{\sqrt{1 - z_3^2}} \left(1 - \frac{z_1}{z_2z_3} + \frac{3}{2}\epsilon + \frac{\epsilon}{1 - z_3^2}\right), \quad (9.3.41)$$

$$U_{32} \simeq \frac{1}{N_2}[(1 - \epsilon)^2 - z_3^2] \simeq \frac{\sqrt{1 - z_3^2}}{\sqrt{2}} \left(1 - \frac{\epsilon}{1 - z_3^2} + \frac{3}{2}\epsilon\right), \quad (9.3.42)$$

$$N_3^2 \simeq (12\epsilon^2 - 1)(4\epsilon^2 - z_3^2) \simeq z_3^2, \quad (9.3.43)$$

$$U_{13} \simeq \frac{1}{N_3}(-2\epsilon z_2 + z_1z_3) \simeq \left(z_1 - 2\epsilon \frac{z_2}{z_3}\right), \quad (9.3.44)$$

$$U_{23} \simeq \frac{1}{N_3}(-2\varepsilon z_1 + z_2 z_3) \simeq z_2, \quad (9.3.45)$$

$$U_{33} \simeq \frac{1}{N_3}(4\varepsilon^2 - z_3^2) \simeq -z_3. \quad (9.3.46)$$

### 9.3.3 Deviation from the bimaximal mixing

Since the parameter value  $\varepsilon$ , (9.3.22), is very small, the Zee mass matrix gives an almost bimaximal mixing. The deviations from the bimaximal mixing are given as follows.

The quantity  $\sin^2 2\theta_{atm}$  in the  $\nu_\mu \rightarrow \nu_\mu$  disappearance experiments is given by

$$\sin^2 2\theta_{atm} = 4|U_{23}|^2|U_{33}|^2. \quad (9.3.47)$$

For a small  $\varepsilon$ , we obtain

$$U_{23}U_{33} = \frac{1}{N_3^2}(z_1\lambda_3 + z_2z_3)(\lambda_3^2 - z_3^2) = \frac{z_1\lambda_3 + z_2z_3}{3\lambda_3^2 - 1} \simeq -z_2z_3(1 + 2a\varepsilon^2 + O(\varepsilon^4)), \quad (9.3.48)$$

and

$$\sin^2 2\theta_{atm} \simeq z_2^2 z_3^2 (1 + 4a\varepsilon^2), \quad (9.3.49)$$

where

$$a = 6 - \frac{1}{z_2^2 z_3^2}. \quad (9.3.50)$$

Since

$$z_2^2 + z_3^2 = 1 - z_1^2, \quad (9.3.51)$$

$z_2^2$  and  $z_3^2$  are solutions of the following equation

$$x^2 - (1 - z_1^2)x + \frac{1}{4}\sin^2 2\theta_{atm}(1 - 4a\varepsilon^2) = 0. \quad (9.3.52)$$

The solutions are given by

$$x = \frac{1}{2}[1 - z_1^2 \pm \sqrt{(1 - z_1^2)^2 - \sin^2 2\theta_{atm}(1 - 4a\varepsilon^2)}]. \quad (9.3.53)$$

Since the part in the square root must be positive, we obtain the constraint

$$\sin^2 2\theta_{atm} \leq \frac{(1 - z_1^2)^2}{1 - 4a\varepsilon^2} \simeq 1 + 6(4 - \frac{1}{z_2^2 z_3^2})\varepsilon^2. \quad (9.3.54)$$

In other words, since we know that the observed value of  $\sin^2 2\theta_{atm}$  is highly close to 1,  $\sin^2 2\theta_{atm} \simeq 1$ , the value  $z_2^2$  must highly be close to  $z_3^2$ . By neglecting  $O(\varepsilon^2)$ , we estimate

$$(z_3^2 \text{ or } z_2^2) \simeq \frac{1}{2}(1 \pm \sqrt{1 - \sin^2 2\theta_{atm}}). \quad (9.3.55)$$

On the other hand,  $\sin^2 2\theta_{solar}$  is given by

$$\sin^2 2\theta_{solar} = 4|U_{11}|^2|U_{12}|^2. \quad (9.3.56)$$

Since

$$U_{11}U_{12} = \frac{1}{N_1 N_2}(z_2\lambda_1 + z_1z_3)(z_2\lambda_2 + z_1z_3)$$

$$= \frac{z_1^2 z_3^2 - z_1 z_2 z_3 \lambda_3 + z_2^2 \lambda_1 \lambda_2}{\sqrt{(3\lambda_1^2 - 1)(3\lambda_2^2 - 1)(\lambda_1^2 - z_3^2)(\lambda_2^2 - z_3^2)}}, \quad (9.3.57)$$

by taking the approximations (9.3.15)-(9.3.17), we obtain

$$U_{11}U_{12} \simeq -\frac{z_2^2 - (\frac{1}{z_2} + 2 + 4z_2^2)\varepsilon^2}{2(1 - z_3^2)\sqrt{1 - \frac{23 - 34z_2^2 + 15z_3^4}{(1 - z_3^2)^2}\varepsilon^2}}$$

$$\simeq \frac{z_2^2}{2(z_2^2 + z_1^2)} \left[ 1 - \frac{1}{z_2^4}(1 + 2z_2^2 + 4z_2^4)\varepsilon^2 + \frac{1}{2} \frac{23 - 34z_2^2 + 15z_3^4}{(1 - z_3^2)^2}\varepsilon^2 \right] \simeq -\frac{1}{2} \left( 1 - \frac{1}{2}\varepsilon^2 \right), \quad (9.3.58)$$

where we have used  $z_2^2 \simeq \frac{1}{2} + O(\varepsilon^2)$  and  $z_3^2 \simeq \frac{1}{2} + O(\varepsilon^2)$ . Therefore, we obtain

$$\sin^2 2\theta_{solar} \simeq 1 - \varepsilon^2 \simeq 1 - \frac{1}{16} \left( \frac{\Delta m_{solar}^2}{\Delta m_{atm}^2} \right)^2. \quad (9.3.59)$$

Even we take the relation (9.4.55) into consideration, we obtain <sup>10</sup>

$$\sin^2 2\theta_{solar} \geq 1 - \frac{1}{16} \left( \frac{\Delta m_{solar}^2}{\Delta m_{atm}^2} \right)^2. \quad (9.3.60)$$

### 9.3.4 Case with $z_2 = z_3$

For the case with  $z_2 = z_3$ , since  $2z_3^2 + z_1^2 = 1$ , we obtain

$$\varepsilon = z_1 z_2 z_3 = z_3^2 z_1 = \frac{1}{2} z_1 (1 - z_1^2), \quad (9.3.61)$$

so that we obtain the mixing matrix

$$U \simeq \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \end{pmatrix}. \quad (9.3.62)$$

In fact, we obtain the exact expression for  $z_2 = z_3$  is as follow:

$$\lambda_1 = \sqrt{1 - \frac{3}{4}z_1^2} + \frac{1}{2}z_1, \quad (9.3.63)$$

$$\lambda_2 = -\left( \sqrt{1 - \frac{3}{4}z_1^2} - \frac{1}{2}z_1 \right), \quad (9.3.64)$$

$$\lambda_3 = -z_1, \quad (9.3.65)$$

$$U = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \frac{1}{\sqrt{2}} \sin \theta & \frac{1}{\sqrt{2}} \cos \theta & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \sin \theta & \frac{1}{\sqrt{2}} \cos \theta & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad (9.3.66)$$

$$\tan \theta = \sqrt{\frac{\sqrt{1 - 3z_1^2/4} + z_1/2}{\sqrt{1 - 3z_1^2/4} - z_1/2}} = \sqrt{\frac{m_1}{-m_2}}. \quad (9.3.67)$$

<sup>10</sup>Y. Koide, Phys. Rev. **D64**, 077301 (2001).

## 9.4 Matrices which give a nearly bimaximal mixing

The mixing matrix

$$U = \begin{pmatrix} c & s & 0 \\ -\frac{1}{\sqrt{2}}s & \frac{1}{\sqrt{2}}c & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}}s & \frac{1}{\sqrt{2}}c & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad (9.4.1)$$

is called a “nearly bimaximal” mixing, where  $c = \cos \theta$  and  $s = \sin \theta$ .

We will demonstrate examples of such a mass matrix below.

### 9.4.1 Mass matrix with a $2 \leftrightarrow 3$ symmetry

The Hermitian mass matrix

$$M = \begin{pmatrix} d & ae^{i\phi} & ae^{i\phi} \\ ae^{-i\phi} & b & c \\ ae^{-i\phi} & c & b \end{pmatrix}, \quad (9.4.2)$$

is invariant under a  $2 \leftrightarrow 3$  exchange, where  $a, b, \dots$  are real.

The eigenvalues are gives as fallows:

$$\begin{aligned} m_1^2 &= \frac{1}{2}[b + c + d - \sqrt{8a^2 + (b + c - d)^2}], \\ m_2^2 &= \frac{1}{2}[b + c + d + \sqrt{8a^2 + (b + c - d)^2}], \\ m_3^2 &= b - c + d. \end{aligned} \quad (9.4.3)$$

The mixing angle  $\theta$  in Eq.(9.4.1) is given by

$$s = \sqrt{\frac{d - m_1}{m_2 - m_1}}, \quad c = \sqrt{\frac{m_2 - d}{m_2 - m_1}}. \quad (9.4.4)$$

### 9.4.2 Two rank-1 matrix

The matrix which is consisted of two rank-1 matrices

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}(1+x) \begin{pmatrix} a^2 & a & a \\ a & 1 & 1 \\ a & 1 & 1 \end{pmatrix}, \quad (9.4.5)$$

also induces a nearly bimaximal mixing (9.4.1).

The eigenvalues are as follows:

$$\begin{aligned} m_1 &= \frac{1}{4} \left( 4 + 2x + a^2(1+x) - \sqrt{A} \right), \\ m_2 &= \frac{1}{4} \left( 4 + 2x + a^2(1+x) + \sqrt{A} \right), \\ m_3 &= 0, \end{aligned} \quad (9.4.6)$$

where

$$A = 4x^2 + 4a^2(1+x)(2+x) + a^4(1+x)^2. \quad (9.4.7)$$

The mixing angle  $\theta$  in Eq. (9.4.1) is also given by the relation (9.4.4).

## 9.5 Matrices which give a tribimaximal mixing

The mixing matrix

$$U_{TB} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{3}} & 0 \\ 0 & 0 & 1 \end{pmatrix} = R_1(-\frac{\pi}{4})R_3(\phi), \quad (9.5.1)$$

is called a ‘‘tribimaximal’’ mixing<sup>11</sup>, where  $\phi = \sin^{-1}(1/\sqrt{3})$ , and  $R_1$  and  $R_3$  are defined in Eq.(8.1.9).

### 9.5.1 He-Zee matrix

The matrix form<sup>12</sup>

$$M = \begin{pmatrix} 2+x & 0 & 0 \\ 0 & 1-y+x & 1+y \\ 0 & 1+y & 1-y+x \end{pmatrix} + \varepsilon \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad (9.5.2)$$

is proposed by He and Zee as a mass matrix which gives a tribimaximal mixing (9.5.1). The matrix (9.5.1) can also be expressed as

$$M = a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + b \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \varepsilon \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad (9.5.3)$$

with  $a = 1 + y$  and  $b = 1 - y + x$ .

The eigenvalues of the matrix (9.5.2) [and also (9.5.3)] are given as follows:

$$\begin{aligned} m_1 &= 2 + x = a + b, \\ m_2 &= 2 + x + 3\varepsilon = a + b + 3\varepsilon, \\ m_3 &= x - 2y = a - b. \end{aligned} \quad (9.5.4)$$

### 9.5.2 Tribimaximal mixing and $S_3$ symmetry

Generally, doublet  $(\psi_\pi, \psi_\eta)$  and singlet  $\psi_\sigma$  in a permutation symmetry  $S_3$  are defined by

$$\begin{pmatrix} \psi_\pi \\ \psi_\eta \\ \psi_\sigma \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}. \quad (9.5.5)$$

If the charged lepton mass matrix  $M_e$  is diagonal on the basis  $(\psi_1, \psi_2, \psi_3) = (\psi_e, \psi_\mu, \psi_\tau)$ , while the neutrino mass matrix  $\widetilde{M}_\nu \equiv D_\nu$  is diagonal on the  $(\psi_\pi, \psi_\eta, \psi_\sigma)$  basis, the neutrino mass matrix  $M_\nu$  on the  $(\psi_\pi, \psi_\eta, \psi_\sigma)$  basis is diagonalized as follows:

$$U^T M_\nu U = D_\nu \equiv \text{diag}(m_\pi, m_\eta, m_\sigma), \quad (9.5.6)$$

<sup>11</sup>P.F. Harrison, D.H. Perkins, and W.G. Scott, Phys. Lett. **B458**, 79 (1999).

<sup>12</sup>X.-G. He and A. Zee, Phys. Lett. **B560**, 87 (2003).

where  $U$  is given by

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}, \quad (9.5.7)$$

for the definition (9.5.5).

However, if the  $S_3$  eigenstates in the neutrino sector are given by

$$\begin{pmatrix} \psi_\pi \\ \psi_\eta \\ \psi_\sigma \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \psi_3 \\ \psi_1 \\ \psi_2 \end{pmatrix} \quad (9.5.8)$$

where  $(\psi_1, \psi_2, \psi_3) = (\psi_e, \psi_\mu, \psi_\tau)$  in the charged lepton sector, the mixing matrix  $U$  in Eq. (9.5.5) on the  $(\psi_\pi, \psi_\eta, \psi_\sigma)$  basis is given by

$$U = \begin{pmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}. \quad (9.5.9)$$

When we rearrange the eigenvalues as  $D \equiv \text{diag}(m_\eta, m_\sigma, m_\pi)$  for the case of  $|m_\eta| < |m_\sigma| < |m_\pi|$ , we will obtain the mixing matrix

$$U_{TB} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \quad (9.5.10)$$

The form  $M_\nu$  for  $U_\nu = U_{TB}$  is given by

$$M_\nu = U_{TB} \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} U_{TB}^T$$

$$= \frac{m_1}{6} \begin{pmatrix} 4 & -2 & -2 \\ -2 & 1 & 1 \\ -2 & 1 & 1 \end{pmatrix} + \frac{m_2}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \frac{m_3}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$= \frac{m_1}{2} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} + \frac{m_2 - m_1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \frac{m_3}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \quad (9.5.11)$$

$$= m_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{m_2 - m_1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \frac{m_3 - m_1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}. \quad (9.5.12)$$

(Also, see Table 10.5.1.)

## 10 $S_3$ Symmetry and its Related Matrices

The observed neutrino mixing is approximately given by the tribimaximal mixing (9.5.9) which is closely related to an  $S_3$  symmetry as discussed in Sec.9.2.2. In this chapter, we will give more details of the matrices related to the  $S_3$  symmetry.

### 10.1 Doublet and singlet of $S_3$

The  $S_3$  doublet  $(\psi_\pi, \psi_\eta)$  and singlet  $\psi_\sigma$  on a real basis are defined as follows:

$$\begin{pmatrix} \psi_\pi \\ \psi_\eta \\ \psi_\sigma \end{pmatrix} = A \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad (10.1.1)$$

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}. \quad (10.1.2)$$

Hereafter, we will use the following notations

$$\psi_{123} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad \psi_{\pi\eta\sigma} = \begin{pmatrix} \psi_\pi \\ \psi_\eta \\ \psi_\sigma \end{pmatrix}. \quad (10.1.3)$$

We can construct the following scalars from fields  $\phi_a$  and  $\psi_a$  ( $a = \pi, \eta, \sigma$ ):

$$\phi_\sigma \psi_\sigma = \phi_{123}^T X \psi_{123}, \quad (10.1.4)$$

$$\phi_\pi \psi_\pi + \phi_\eta \psi_\eta = \phi_{123}^T (\mathbf{1} - X) \psi_{123}, \quad (10.1.5)$$

$$\phi_\pi \psi_\pi + \phi_\eta \psi_\eta + \phi_\sigma \psi_\sigma = \phi_{123}^T \mathbf{1} \psi_{123}, \quad (10.1.6)$$

where

$$\mathbf{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad X = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \quad (10.1.7)$$

### 10.2 $S_3$ -invariant Yukawa interactions

We consider  $S_3$ -invariant terms which are constructed by  $\bar{\psi} \otimes \phi \otimes \chi$ . We have the following 5 types of the  $S_3$ -invariant terms:

**Type (a)**

$$L_a = \bar{\psi}_\sigma \phi_\sigma \chi_\sigma = \frac{\phi_1 + \phi_2 + \phi_3}{3\sqrt{3}} \bar{\psi}_{123} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \chi_{123}. \quad (10.2.1)$$

**Type (b)**

$$L_b = (\bar{\psi}_\pi \chi_\pi + \bar{\psi}_\eta \chi_\eta) \phi_\sigma = \frac{\phi_1 + \phi_2 + \phi_3}{3\sqrt{3}} \bar{\psi}_{123} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \chi_{123}. \quad (10.2.2)$$

**Type (c)**

$$\begin{aligned} L_c &= (\bar{\psi}_\pi \chi_\eta + \bar{\psi}_\eta \chi_\pi) \phi_\pi + (\bar{\psi}_\pi \chi_\pi - \bar{\psi}_\eta \chi_\eta) \phi_\eta \\ &= \frac{\sqrt{2}}{3\sqrt{3}} \bar{\psi}_{123} \left[ \phi_1 \begin{pmatrix} 2 & -1 & -1 \\ -1 & -1 & 2 \\ -1 & 2 & -1 \end{pmatrix} + \phi_2 \begin{pmatrix} -1 & -1 & 2 \\ -1 & 2 & -1 \\ 2 & -1 & -1 \end{pmatrix} + \phi_3 \begin{pmatrix} -1 & 2 & -1 \\ 2 & -1 & -1 \\ -1 & -1 & 2 \end{pmatrix} \right] \chi_{123}. \end{aligned} \quad (10.2.3)$$

**Type (d)**

$$\begin{aligned} L_d &= \bar{\psi}_\sigma (\phi_\pi \chi_\pi + \phi_\eta \chi_\eta) \\ &= \frac{1}{3\sqrt{3}} \bar{\psi}_{123} \left[ \phi_1 \begin{pmatrix} 2 & -1 & -1 \\ 2 & -1 & -1 \\ 2 & -1 & -1 \end{pmatrix} + \phi_2 \begin{pmatrix} -1 & 2 & -1 \\ -1 & 2 & -1 \\ -1 & 2 & -1 \end{pmatrix} + \phi_3 \begin{pmatrix} -1 & -1 & 2 \\ -1 & -1 & 2 \\ -1 & -1 & 2 \end{pmatrix} \right] \chi_{123}. \end{aligned} \quad (10.2.4)$$

**Type (e)**

$$\begin{aligned} L_e &= (\bar{\psi}_\pi \phi_\pi + \bar{\psi}_\eta \phi_\eta) \chi_\sigma \\ &= \frac{1}{3\sqrt{3}} \bar{\psi}_{123} \left[ \phi_1 \begin{pmatrix} 2 & 2 & 2 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix} + \phi_2 \begin{pmatrix} -1 & -1 & -1 \\ 2 & 2 & 2 \\ -1 & -1 & -1 \end{pmatrix} + \phi_3 \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ 2 & 2 & 2 \end{pmatrix} \right] \chi_{123}. \end{aligned} \quad (10.2.5)$$

On the other hand, if we start from the  $\psi_{123}$  base, we can consider the following two typical  $S_3$ -invariant terms:

**Type (A)**

$$L_A = \sum_i^3 \bar{\psi}_i \phi_i \chi_i = \bar{\psi}_{123} \begin{pmatrix} \phi_1 & 0 & 0 \\ 0 & \phi_2 & 0 \\ 0 & 0 & \phi_3 \end{pmatrix} \chi_{123} = \bar{\psi}_{\pi\eta\sigma} M_A \chi_{\pi\eta\sigma}, \quad (10.2.6)$$

where

$$M_A = A \begin{pmatrix} \phi_1 & 0 & 0 \\ 0 & \phi_2 & 0 \\ 0 & 0 & \phi_3 \end{pmatrix} A^T = \frac{1}{\sqrt{3}} \begin{pmatrix} \phi_\sigma + \frac{1}{\sqrt{2}} \phi_\eta & \frac{1}{\sqrt{2}} \phi_\pi & \phi_\pi \\ \frac{1}{\sqrt{2}} \phi_\pi & \phi_\sigma - \frac{1}{\sqrt{2}} \phi_\eta & \phi_\eta \\ \phi_\pi & \phi_\eta & \phi_\sigma \end{pmatrix}. \quad (10.2.7)$$

**Type (B)**

$$L_B = \sum_{i,j,k:\text{cyclic}} (\bar{\psi}_i \phi_j \chi_k + \bar{\psi}_k \phi_j \chi_i) = \bar{\psi}_{123} \begin{pmatrix} 0 & \phi_3 & \phi_2 \\ \phi_3 & 0 & \phi_1 \\ \phi_2 & \phi_1 & 0 \end{pmatrix} \chi_{123} = \bar{\psi}_{\pi\eta\sigma} M_B \chi_{\pi\eta\sigma}, \quad (10.2.8)$$

where

$$M_B = A \begin{pmatrix} 0 & \phi_3 & \phi_2 \\ \phi_3 & 0 & \phi_1 \\ \phi_2 & \phi_1 & 0 \end{pmatrix} A^T = \frac{1}{\sqrt{3}} \begin{pmatrix} -\phi_\sigma + \sqrt{2}\phi_\eta & \sqrt{2}\phi_\pi & -\phi_\pi \\ \sqrt{2}\phi_\pi & -\phi_\sigma - \sqrt{2}\phi_\eta & -\phi_\eta \\ -\phi_\pi & -\phi_\eta & 2\phi_\sigma \end{pmatrix}. \quad (10.2.9)$$

The interactions  $L_A$  and  $L_B$  are explicitly given by

$$L_A = \frac{1}{\sqrt{3}} \left[ (\bar{\psi}_\pi \chi_\pi + \bar{\psi}_\eta \chi_\eta + \bar{\psi}_\sigma \chi_\sigma) \phi_\sigma + (\bar{\psi}_\pi \phi_\pi + \bar{\psi}_\eta \phi_\eta) \chi_\sigma + \bar{\psi}_\sigma (\phi_\pi \chi_\pi + \phi_\eta \chi_\eta) \right. \\ \left. + \frac{1}{\sqrt{2}} (\bar{\psi}_\pi \phi_\pi \chi_\eta + \bar{\psi}_\eta \phi_\pi \chi_\pi + \bar{\psi}_\pi \phi_\eta \chi_\pi - \bar{\psi}_\eta \phi_\eta \chi_\eta) \right], \quad (10.2.10)$$

$$L_B = \frac{1}{\sqrt{3}} \left[ -(\bar{\psi}_\pi \chi_\pi + \bar{\psi}_\eta \chi_\eta + 2\bar{\psi}_\sigma \chi_\sigma) \phi_\sigma - (\bar{\psi}_\pi \phi_\pi + \bar{\psi}_\eta \phi_\eta) \chi_\sigma - \bar{\psi}_\sigma (\phi_\pi \chi_\pi + \phi_\eta \chi_\eta) \right. \\ \left. + \sqrt{2} (\bar{\psi}_\pi \phi_\pi \chi_\eta + \bar{\psi}_\eta \phi_\pi \chi_\pi + \bar{\psi}_\pi \phi_\eta \chi_\pi - \bar{\psi}_\eta \phi_\eta \chi_\eta) \right]. \quad (10.2.11)$$

If the mass matrix  $M$  for the  $\psi_{\pi\eta\sigma}$  basis is given by  $aM_A + bM_B$ , the matrix  $M$  is expressed as

$$M = aM_A + bM_B = \frac{1}{\sqrt{3}} \begin{pmatrix} \frac{a+2b}{\sqrt{2}}\phi_\eta & \frac{a+2b}{\sqrt{2}}\phi_\pi & (a-b)\phi_\pi \\ \frac{a+2b}{\sqrt{2}}\phi_\pi & -\frac{a+2b}{\sqrt{2}}\phi_\eta & (a-b)\phi_\eta \\ (a-b)\phi_\pi & (a-b)\phi_\eta & 3b\phi_\sigma \end{pmatrix} + \frac{a-b}{\sqrt{3}}\phi_\sigma \mathbf{1}, \quad (10.2.12)$$

or

$$M = \frac{a+2b}{\sqrt{6}} \begin{pmatrix} \phi_\eta & \phi_\pi & 0 \\ \phi_\pi & -\phi_\eta & 0 \\ 0 & 0 & \sqrt{2}\phi_\sigma \end{pmatrix} + \frac{a-b}{\sqrt{3}} \begin{pmatrix} \phi_\sigma & 0 & \phi_\pi \\ 0 & \phi_\sigma & \phi_\eta \\ \phi_\pi & \phi_\eta & 0 \end{pmatrix}. \quad (10.2.13)$$

### 10.3 Democratic seesaw model

We assume the  $S_3$ -invariant Yukawa interactions

$$L_Y = y_L \Sigma_i \bar{q}_{Li} \phi_{Li} Q_{Ri} + y_R \bar{Q}_{Li} \phi_{Ri} q_{Ri} + \bar{Q}_L M_Q Q_R + h.c. \quad (10.3.1)$$

We also assume<sup>13</sup> that the mass matrix  $M_Q$  has a diagonal form on the  $Q_{\pi\eta\sigma}$  basis, i.e.

$$\bar{Q}_{L\pi\eta\sigma} M_Q Q_{R\pi\eta\sigma} = \bar{Q}_{L\pi\eta\sigma} \begin{pmatrix} M_D & 0 & 0 \\ 0 & M_D & 0 \\ 0 & 0 & M_S \end{pmatrix} Q_{R\pi\eta\sigma}. \quad (10.3.2)$$

Then, when we define the effective mass matrix

$$\bar{q}_{L\pi\eta\sigma} M_{\pi\eta\sigma} q_{R\pi\eta\sigma}, \quad (10.3.3)$$

<sup>13</sup>The assumption (10.3.2) is incompatible with the assumption  $\bar{Q}_{L123} = \text{diag}(M_1, M_2, M_3) Q_{R123}$ , which leads to the non-diagonal form of  $M_Q$  similar to the expression (10.2.7).

we obtain the following mass matrix form

$$\begin{aligned}
M_{\pi\eta\sigma} &= y_L y_R A D_L A^T \begin{pmatrix} M_D^{-1} & 0 & 0 \\ 0 & M_D^{-1} & 0 \\ 0 & 0 & M_S^{-1} \end{pmatrix} A D_R A^T \\
&= y_L y_R A D_L \left( \frac{1}{M_D} (\mathbf{1} - X) + \frac{1}{M_S} X \right) D_R A^T,
\end{aligned} \tag{10.3.4}$$

where

$$\begin{aligned}
D_L &= \text{diag}(v_{L_1}, v_{L_2}, v_{L_3}), \\
D_R &= \text{diag}(v_{R_1}, v_{R_2}, v_{R_3}),
\end{aligned} \tag{10.3.5}$$

and  $v_i = \langle \phi_i \rangle$ . Therefore, we obtain a democratic seesaw-model form on the  $q_{123}$  basis,

$$M_{123} = \frac{y_L y_R}{M_D} D_L (\mathbf{1} + aX) D_R, \tag{10.3.6}$$

with

$$a = \frac{M_D}{M_S} - 1. \tag{10.3.7}$$

For the  $q_{\pi\eta\sigma}$  basis, since

$$A(\mathbf{1} - X)A^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{10.3.8}$$

$$AXA^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{10.3.9}$$

we obtain the following mass matrix form

$$\begin{aligned}
M_{\pi\eta\sigma} &= y_L y_R A D_L A^T \left[ \frac{1}{M_D} A(\mathbf{1} - X)A^T + \frac{1}{M_S} AXA^T \right] A D_R A^T \\
&= \frac{y_L y_R}{3M_D} \begin{pmatrix} (v_{L\sigma} + \frac{1}{\sqrt{2}}v_{L\eta})(v_{R\sigma} + \frac{1}{\sqrt{2}}v_{R\eta}) + \frac{1}{2}v_{L\pi}v_{R\pi} \\ \frac{1}{\sqrt{2}}v_{L\pi}(v_{R\sigma} + \frac{1}{\sqrt{2}}v_{R\eta}) + \frac{1}{\sqrt{2}}(v_{L\sigma} - \frac{1}{\sqrt{2}}v_{L\eta})v_{R\pi} \\ v_{L\pi}(v_{R\sigma} + \frac{1}{\sqrt{2}}v_{R\eta}) + \frac{1}{\sqrt{2}}v_{L\eta}v_{R\pi} \end{pmatrix} \\
&\quad \begin{pmatrix} \frac{1}{\sqrt{2}}(v_{L\sigma} + \frac{1}{\sqrt{2}}v_{L\eta})v_{R\pi} + \frac{1}{\sqrt{2}}v_{L\pi}(v_{R\sigma} - \frac{1}{\sqrt{2}}v_{R\eta}) & (v_{L\sigma} + \frac{1}{\sqrt{2}}v_{L\eta})v_{R\pi} + \frac{1}{\sqrt{2}}v_{L\pi}v_{R\eta} \\ \frac{1}{2}v_{L\pi}v_{R\pi} + (v_{L\sigma} - \frac{1}{\sqrt{2}}v_{L\eta})(v_{R\sigma} - \frac{1}{\sqrt{2}}v_{R\eta}) & \frac{1}{\sqrt{2}}v_{L\pi}v_{R\eta} + (v_{L\sigma} - \frac{1}{\sqrt{2}}v_{L\eta})v_{R\eta} \\ \frac{1}{\sqrt{2}}v_{L\pi}v_{R\pi} + v_{L\eta}(v_{R\sigma} - \frac{1}{\sqrt{2}}v_{R\eta}) & v_{L\pi}v_{R\pi} + v_{L\eta}v_{R\eta} \end{pmatrix} \\
&\quad + \frac{y_L y_R}{3M_S} \begin{pmatrix} v_{L\pi}v_{R\pi} & v_{L\pi}v_{R\eta} & v_{L\pi}v_{R\sigma} \\ v_{L\eta}v_{R\pi} & v_{L\eta}v_{R\eta} & v_{L\eta}v_{R\sigma} \\ v_{L\sigma}v_{R\pi} & v_{L\sigma}v_{R\eta} & v_{L\sigma}v_{R\sigma} \end{pmatrix}.
\end{aligned} \tag{10.3.10}$$

If we assume  $v_{L_i} = v_{R_i} \equiv v_i$ , we obtain

$$M_{\pi\eta\sigma} = \frac{y_L y_R}{3M_D} \begin{pmatrix} v_\sigma^2 + \frac{1}{2}(v_\pi^2 + v_\eta^2) + \sqrt{2}v_\sigma v_\eta \\ \sqrt{2}v_\pi v_\sigma \\ v_\pi v_\sigma + \sqrt{2}v_\pi v_\eta \end{pmatrix}$$

$$\begin{aligned}
& \begin{pmatrix} \sqrt{2}v_\pi v_\sigma & v_\pi v_\sigma + \sqrt{2}v_\pi v_\eta \\ v_\sigma^2 + \frac{1}{2}(v_\pi^2 + v_\eta^2) - \sqrt{2}v_\sigma v_\eta & v_\eta v_\sigma + \frac{1}{\sqrt{2}}(v_\pi^2 - v_\eta^2) \\ v_\eta v_\sigma + \frac{1}{\sqrt{2}}(v_\pi^2 - v_\eta^2) & v_\pi^2 + v_\eta^2 \end{pmatrix} \\
& + \frac{yLYR}{3M_S} \begin{pmatrix} v_\pi^2 & v_\pi v_\eta & v_\pi v_\sigma \\ v_\pi v_\eta & v_\eta^2 & v_\eta v_\sigma \\ v_\pi v_\sigma & v_\eta v_\sigma & v_\sigma^2 \end{pmatrix}. \tag{10.3.11}
\end{aligned}$$

The second term is a rank-1 matrix, whose eigenvalues are  $(0, 0, v_\pi^2 + v_\eta^2 + v_\sigma^2)$ . If the second term is dominant, the matrix is diagonalized by the mixing matrix

$$U = \begin{pmatrix} c_{12} & s_{12}c_{23} & s_{12}s_{23} \\ -s_{12} & c_{12}c_{23} & c_{12}s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} = R_3(\theta_{12})R_1(\theta_{23}), \tag{10.3.12}$$

where

$$s_{12} = \frac{v_\pi}{\sqrt{v_\pi^2 + v_\eta^2}}, \quad c_{12} = \frac{v_\eta}{\sqrt{v_\pi^2 + v_\eta^2}}. \tag{10.3.13}$$

$$s_{23} = \sqrt{\frac{v_\pi^2 + v_\eta^2}{v_\pi^2 + v_\eta^2 + v_\sigma^2}}, \quad c_{23} = \frac{v_\sigma}{\sqrt{v_\pi^2 + v_\eta^2 + v_\sigma^2}}. \tag{10.3.14}$$

**Table 10.3.1** Matrix forms on the bases  $\psi_{123}$  and  $\psi_{\pi\eta\sigma}$

(1, 2, 3) basis	$(\pi, \eta, \sigma)$ basis
$\begin{pmatrix} v_1 & 0 & 0 \\ 0 & v_2 & 0 \\ 0 & 0 & v_3 \end{pmatrix}$	$\frac{1}{\sqrt{3}} \begin{pmatrix} v_\sigma + \frac{1}{\sqrt{2}}v_\eta & \frac{1}{\sqrt{2}}v_\pi & v_\pi \\ \frac{1}{\sqrt{2}}v_\pi & v_\sigma - \frac{1}{\sqrt{2}}v_\eta & v_\eta \\ v_\pi & v_\eta & v_\sigma \end{pmatrix}$
$\begin{pmatrix} 0 & v_3 & v_2 \\ v_3 & 0 & v_1 \\ v_2 & v_1 & 0 \end{pmatrix}$	$\frac{1}{\sqrt{3}} \begin{pmatrix} -v_\sigma + \sqrt{2}v_\eta & \sqrt{2}v_\pi & -v_\pi \\ \sqrt{2}v_\pi & -v_\sigma - \sqrt{2}v_\eta & -v_\eta \\ -v_\pi & -v_\eta & 2v_\sigma \end{pmatrix}$
$\frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$\frac{1}{3\sqrt{2}} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$	$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$-\frac{1}{\sqrt{6}} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$	$\frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$
$\frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$	$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$\frac{1}{3\sqrt{2}} \begin{pmatrix} 1 & -2 & 1 \\ -2 & 1 & 1 \\ 1 & 1 & -2 \end{pmatrix}$	$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$M_S^{-1} D_L (\mathbf{1} + aX) D_R$ $a = \frac{M_D}{M_S} - 1$	$\tilde{D}_L \begin{pmatrix} M_D^{-1} & 0 & 0 \\ 0 & M_D^{-1} & 0 \\ 0 & 0 & M_S^{-1} \end{pmatrix} \tilde{D}_R$ $\tilde{D} = ADA^T$

## 10.4 Dual bases $A$ and $B$

We define bases  $A$  and  $B$  as follows:

$$\begin{pmatrix} \psi_\pi^A \\ \psi_\eta^A \\ \psi_\sigma^A \end{pmatrix} = A \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad \begin{pmatrix} \psi_\pi^B \\ \psi_\eta^B \\ \psi_\sigma^B \end{pmatrix} = B \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad (10.4.1)$$

where

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}. \quad (10.4.2)$$

The basis  $A$  is known as a convention of the singlet  $\psi_\sigma$  and doublet  $(\psi_\pi, \psi_\eta)$  of  $S_3$ . The mixing matrix  $B$  is related to the so-called ‘‘tribimaximal mixing’’ in the neutrino mixing as we discussed in Sec.8.2.2.

The basis  $A$  is connected by the matrix

$$U_{AB} \equiv AB^T = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (10.4.3)$$

as

$$\begin{pmatrix} \psi_\pi^A \\ \psi_\eta^A \\ \psi_\sigma^A \end{pmatrix} = U_{AB} \begin{pmatrix} \psi_\eta^B \\ \psi_\sigma^B \\ \psi_\pi^B \end{pmatrix}. \quad (10.4.4)$$

Since

$$U_{AB}^2 = \mathbf{1}, \quad (10.4.5)$$

It can be regarded that the bases  $A$  and  $B$  are dual each other. The  $S_3$ -invariant interactions (see Sec.9.2) are, of course, invariant under the transformation  $U_{AB}$ .

Under the definition of the basis  $B$ , the tribimaximal mixing is given by

$$U = \begin{pmatrix} -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad (10.4.6)$$

instead of Eq.(9.5.1).

However, note that if we define another  $S_3$  basis  $B'$  (see Eq.(9.5.7))

$$\begin{pmatrix} \psi_\pi^{B'} \\ \psi_\eta^{B'} \\ \psi_\sigma^{B'} \end{pmatrix} = B' \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad (10.4.7)$$

the transformation

$$U_{AB'} = A(B')^T = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (10.4.8)$$

does not give  $(U_{AB'})^2 = \mathbf{1}$ :

$$(U_{AB'})^2 = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \neq \mathbf{1}, \quad (10.4.9)$$

$$(U_{AB'})^3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (10.4.10)$$

The  $S_3$ -invariant interactions are not invariant under the transformation  $U_{AB'}$ . Only when we define the basis  $A'$

$$\begin{pmatrix} \psi_{\pi}^{A'} \\ \psi_{\eta}^{A'} \\ \psi_{\sigma}^{A'} \end{pmatrix} = A' \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad (10.4.11)$$

we obtain

$$U_{A'B'} = A'(B')^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (10.4.12)$$

$$(U_{A'B'})^2 = \mathbf{1}. \quad (10.4.13)$$

The  $S_3$ -invariant interactions are invariant under the transformation  $U_{A'B'}$ . Under the definition of the basis  $B'$ , the tribimaximal mixing is expressed by Eq.(9.5.1) instead of Eq.(10.4.6).

## 10.5 Useful basis for neutrino mass matrix

The observed neutrino mixing is very close to the so-called tribimaximal mixing<sup>14</sup>

$$U_{TB} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{2} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{2} \end{pmatrix}. \quad (10.5.1)$$

Therefore, for investigation of the neutrino mass matrix, the definition

$$\begin{pmatrix} \psi_{\eta} \\ \psi_{\sigma} \\ \psi_{\pi} \end{pmatrix} = U_{TB}^T \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad (10.5.2)$$

is useful rather than the definition (10.1.1). This means that if the charged lepton mass eigenstates ( $e^-$ ,  $\mu^-$ ,  $\tau^-$ ) are given by the basis  $(\psi_1, \psi_2, \psi_3)$ , then the neutrino mass eigenstates are  $(\psi_{\eta}, \psi_{\sigma}, \psi_{\pi})$  with mass hierarchy  $m_{\eta}^2 < m_{\sigma}^2 \ll m_{\pi}^2$  (or  $m_{\pi}^2 \ll m_{\eta}^2 < m_{\sigma}^2$ ).

Although the examples of matrix forms  $U_{TB}^T M U_{TB}$  for matrices  $M$  have been essentially given by Table 6.2.1 (Sec.6.2) and Table 10.3.1 (Sec.10.3), we again give some of them under the new basis (10.4.15) in Table 10.5.1.

<sup>14</sup>The phase convention (10.4.14) is identical with (9.5.1), but it is different from one given by (10.4.6).

**Table 10.5.1** Matrix form on the  $(\eta, \sigma, \pi)$  basis

(1, 2, 3) basis	( $\eta, \sigma, \pi$ ) basis	
$M$	$U_{TB}^T M U_{TB}$	
$\frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	(10.5.3)
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	(10.5.4)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$	(10.5.5)
$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	(10.5.6)
$\frac{1}{\sqrt{6}} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\frac{1}{\sqrt{6}} \begin{pmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	(10.5.7)
$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\frac{1}{\sqrt{6}} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & \sqrt{2} \\ -1 & \sqrt{2} & 0 \end{pmatrix}$	(10.5.8)

If we define  $\Phi$

$$\Phi = \text{diag}(\phi_{11}, \phi_{22}, \phi_{33}), \quad (10.5.9)$$

where

$$\begin{aligned} \phi_{11} &= \frac{1}{\sqrt{3}}\phi_\sigma + \frac{2}{\sqrt{6}}\phi_\eta, \\ \phi_{22} &= \frac{1}{\sqrt{3}}\phi_\sigma - \frac{1}{\sqrt{6}}\phi_\eta - \frac{1}{\sqrt{2}}\phi_\pi, \\ \phi_{33} &= \frac{1}{\sqrt{3}}\phi_\sigma - \frac{1}{\sqrt{6}}\phi_\eta + \frac{1}{\sqrt{2}}\phi_\pi, \end{aligned} \quad (10.5.10)$$

we obtain

$$U_{TB}^T \Phi U_{TB} = \frac{1}{\sqrt{3}} \begin{pmatrix} \phi_\sigma + \frac{1}{\sqrt{2}}\phi_\eta & \phi_\eta & -\frac{1}{\sqrt{2}}\phi_\pi \\ \phi_\eta & \phi_\sigma & \phi_\pi \\ -\frac{1}{\sqrt{2}}\phi_\pi & \phi_\pi & \phi_\sigma - \frac{1}{\sqrt{2}}\phi_\eta \end{pmatrix}. \quad (10.5.11)$$

## 11 $A_4$ Symmetry and its Related Matrices

The tribimaximal mixing (9.5.9) can also be understood by an  $A_4$  symmetry. In this chapter, we will investigate possible mass matrices under the  $A_4$  invariance.

### 11.1 Triplets and singlets composed of $A_4$ triplets

When we define triplets of  $A_4$  as follows:

$$\mathbf{3} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad (11.1.1)$$

we obtain the following triplets which are composed of  $\mathbf{3} \times \mathbf{3}$

$$(\mathbf{3} \times \mathbf{3})_{\mathbf{3}} = \begin{pmatrix} a_2 b_3 \\ a_3 b_1 \\ a_1 b_2 \end{pmatrix}, \quad \begin{pmatrix} a_3 b_2 \\ a_1 b_3 \\ a_2 b_1 \end{pmatrix}. \quad (11.1.2)$$

We can also obtain  $\mathbf{1}$ ,  $\mathbf{1}'$ ,  $\mathbf{1}''$  from  $\mathbf{3} \times \mathbf{3}$  as follows:

$$\mathbf{1} = \frac{1}{\sqrt{3}}(a_1 b_1 + a_2 b_2 + a_3 b_3), \quad (11.1.3)$$

$$\mathbf{1}' = \frac{1}{\sqrt{3}}(a_1 b_1 + a_2 b_2 \omega + a_3 b_3 \omega^2), \quad (11.1.4)$$

$$\mathbf{1}'' = \frac{1}{\sqrt{3}}(a_1 b_1 + a_2 b_2 \omega^2 + a_3 b_3 \omega), \quad (11.1.5)$$

where

$$\omega = e^{i\frac{2}{3}\pi} = \frac{-1 + i\sqrt{3}}{2}. \quad (11.1.6)$$

When we define

$$(ab)_{\sigma} = \frac{1}{\sqrt{3}}(a_1 b_1 + a_2 b_2 + a_3 b_3), \quad (11.1.7)$$

$$(ab)_{\eta} = \frac{1}{\sqrt{6}}(2a_1 b_1 - a_2 b_2 - a_3 b_3), \quad (11.1.8)$$

$$(ab)_{\pi} = \frac{1}{\sqrt{2}}(a_3 b_3 - a_2 b_2), \quad (11.1.9)$$

Eqs.(11.1.3)-(11.1.5) are expressed as

$$\mathbf{1} = (ab)_{\sigma}, \quad (11.1.10)$$

$$\mathbf{1}' = \frac{1}{\sqrt{2}}[(ab)_{\eta} - i(ab)_{\pi}], \quad (11.1.11)$$

$$\mathbf{1}'' = \frac{1}{\sqrt{2}}[(ab)_{\eta} + i(ab)_{\pi}]. \quad (11.1.12)$$

We can obtain scalars ( $A_4$  invariant terms) as follows:

$$\mathbf{1}' \times \mathbf{1}'' \sim \mathbf{1}, \quad (11.1.13)$$

$$\mathbf{1}' \times \mathbf{1}' \times \mathbf{1}' \sim \mathbf{1}'' \times \mathbf{1}' \sim \mathbf{1}. \quad (11.1.14)$$

## 11.2 Yukawa interactions $(\bar{\psi}\psi)\phi = (\mathbf{3} \times \mathbf{3})_3 \times \mathbf{3}$

For

$$\bar{\psi} = \begin{pmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \\ \bar{\psi}_3 \end{pmatrix}, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \quad (11.2.1)$$

where, although  $\bar{\psi}_i$  are independent of  $\psi_i$ , we have used same notations for simplicity, we obtain the  $A_4$  invariant Yukawa interaction

$$H_{(\mathbf{3},\mathbf{3},\mathbf{3})} = y_1(\bar{\psi}_2\psi_3\phi_1 + \bar{\psi}_3\psi_1\phi_2 + \bar{\psi}_1\psi_2\phi_3) + y_2(\bar{\psi}_3\psi_2\phi_1 + \bar{\psi}_1\psi_3\phi_2 + \bar{\psi}_2\psi_1\phi_3), \quad (11.2.2)$$

so that the mass matrix form on the basis  $(\psi_1, \psi_2, \psi_3)$  is given by

$$M = \begin{pmatrix} 0 & y_1 v_3 & y_2 v_2 \\ y_2 v_3 & 0 & y_1 v_1 \\ y_1 v_2 & y_2 v_1 & 0 \end{pmatrix}, \quad (11.2.3)$$

where  $v_i = \langle \phi_i \rangle$ . When  $y_1 = y_2$ , the mass matrix (11.2.3) is known as the Zee-type mass matrix.

## 11.3 Yukawa interactions $(\bar{\psi}\psi)\phi = (\mathbf{3} \times \mathbf{3})_1 \times (\mathbf{1}, \mathbf{1}', \mathbf{1}'')$

We consider a case

$$\phi = (\mathbf{1}, \mathbf{1}', \mathbf{1}'') = \left( \phi_\sigma, \frac{1}{\sqrt{2}}(\phi_\eta - i\phi_\pi), \frac{1}{\sqrt{2}}(\phi_\eta + i\phi_\pi) \right). \quad (11.3.1)$$

$$\begin{aligned} H_{(\mathbf{3},\mathbf{3},\mathbf{1})} &= y_0(\bar{\psi}\psi)_\sigma\phi_\sigma + y_1\frac{1}{\sqrt{2}}\left[(\bar{\psi}\psi)_\eta - i(\bar{\psi}\psi)_\pi\right]\frac{1}{\sqrt{2}}(\phi_\eta + i\phi_\pi) + y_2\frac{1}{\sqrt{2}}\left[(\bar{\psi}\psi)_\eta + i(\bar{\psi}\psi)_\pi\right]\frac{1}{\sqrt{2}}(\phi_\eta - i\phi_\pi) \\ &= y_0(\bar{\psi}\psi)_\sigma\phi_\sigma + \frac{1}{2}(y_1 + y_2)\left[(\bar{\psi}\psi)_\eta\phi_\eta + (\bar{\psi}\psi)_\pi\phi_\pi\right] + \frac{i}{2}(y_1 - y_2)\left[(\bar{\psi}\psi)_\eta\phi_\pi - (\bar{\psi}\psi)_\pi\phi_\eta\right]. \end{aligned} \quad (11.3.2)$$

Therefore, if we assume the universality of the coupling constants

$$y_0 = y_1 = y_2, \quad (11.3.3)$$

we obtain the mass matrix on the  $(\psi_1, \psi_2, \psi_3)$  basis

$$M = y_0 \text{diag} \left( \frac{1}{\sqrt{3}}v_\sigma + \frac{2}{\sqrt{6}}v_\eta, \frac{1}{\sqrt{3}}v_\sigma - \frac{1}{\sqrt{6}}v_\eta - \frac{1}{\sqrt{2}}v_\pi, \frac{1}{\sqrt{3}}v_\sigma - \frac{1}{\sqrt{6}}v_\eta + \frac{1}{\sqrt{2}}v_\pi \right). \quad (11.3.4)$$

If we define  $\phi_i$  as

$$\begin{pmatrix} \phi_\sigma \\ \phi_\eta \\ \phi_\pi \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \quad (11.3.5)$$

we obtain

$$M = \text{diag}(v_1, v_2, v_3). \quad (11.3.6)$$

Here, note that  $\phi_i$  defined in (11.3.5) are not members  $\phi_i$  of the triplet  $\phi$  defined in (11.2.1).

### 11.4 Yukawa interactions $(\bar{\psi}\psi)\phi = (\mathbf{3} \times (\mathbf{1}, \mathbf{1}', \mathbf{1}'')) \times \mathbf{3}$

For the case  $(\bar{\psi}\psi)\phi = (\mathbf{3} \times (\mathbf{1}, \mathbf{1}', \mathbf{1}'')) \times \mathbf{3}$ , we obtain

$$\begin{aligned} H_{(\mathbf{3}, \mathbf{1}, \mathbf{3})} &= y_0(\bar{\psi}\phi)_1\psi_1 + y_1(\bar{\psi}\phi)_{1'}\psi_{1''} + y_2(\bar{\psi}\phi)_{1''}\psi_{1'} \\ &= y_0(\bar{\psi}\phi)_\sigma\psi_\sigma + \frac{1}{2}(y_1 + y_2) \left[ (\bar{\psi}\phi)_\eta\psi_\eta + (\bar{\psi}\phi)_\pi\psi_\pi \right] + \frac{i}{2}(y_1 - y_2) \left[ (\bar{\psi}\phi)_\eta\psi_\pi - (\bar{\psi}\phi)_\pi\psi_\eta \right]. \end{aligned} \quad (11.4.1)$$

Therefore, when we define the mass matrix  $M$  as

$$H_{(\mathbf{3}, \mathbf{1}, \mathbf{3})} = (\bar{\psi}_1 \ \bar{\psi}_2 \ \bar{\psi}_3)M \begin{pmatrix} \psi_\sigma \\ \psi_\eta \\ \psi_\pi \end{pmatrix}, \quad (11.4.2)$$

we obtain

$$\begin{aligned} M &= \begin{pmatrix} \frac{1}{\sqrt{3}}y_0v_1 & \frac{y_1+y_2}{2}\frac{1}{\sqrt{6}}v_1 & i\frac{y_1-y_2}{2}\frac{1}{\sqrt{6}}v_1 \\ \frac{1}{\sqrt{3}}y_0v_2 & -\left(\frac{y_1+y_2}{2}\frac{1}{\sqrt{6}} - i\frac{y_1-y_2}{2}\frac{1}{\sqrt{2}}\right)v_2 & -\left(\frac{y_1+y_2}{2}\frac{1}{\sqrt{2}} + i\frac{y_1-y_2}{2}\frac{1}{\sqrt{6}}\right)v_2 \\ \frac{1}{\sqrt{3}}y_0v_3 & -\left(\frac{y_1+y_2}{2}\frac{1}{\sqrt{6}} + i\frac{y_1-y_2}{2}\frac{1}{\sqrt{2}}\right)v_3 & \left(\frac{y_1+y_2}{2}\frac{1}{\sqrt{2}} - i\frac{y_1-y_2}{2}\frac{1}{\sqrt{6}}\right)v_3 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{3}}y_0v_1 & \frac{y_1+y_2}{\sqrt{6}}v_1 & i\frac{y_1-y_2}{\sqrt{6}}v_1 \\ \frac{1}{\sqrt{3}}y_0v_2 & -\frac{1}{\sqrt{6}}(y_1e^{-i\frac{\pi}{3}} + y_2e^{i\frac{\pi}{3}})v_2 & -\frac{1}{\sqrt{6}}(y_1e^{i\frac{\pi}{6}} + y_2e^{-i\frac{\pi}{6}})v_2 \\ \frac{1}{\sqrt{3}}y_0v_3 & -\frac{1}{\sqrt{6}}(y_1e^{i\frac{\pi}{3}} + y_2e^{-i\frac{\pi}{3}})v_3 & \frac{1}{\sqrt{6}}(y_1e^{-i\frac{\pi}{6}} + y_2e^{i\frac{\pi}{6}})v_3 \end{pmatrix}. \end{aligned} \quad (11.4.3)$$

If  $y_0 = y_1 = y_2$ , we obtain

$$M = DU_{TB}, \quad (11.4.4)$$

where

$$D = y_0 \text{diag}(v_1, v_2, v_3), \quad (11.4.5)$$

$$U_{TB} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad (11.4.6)$$

on the  $(\psi_\eta, \psi_\sigma, \psi_\pi)$  basis. The mixing matrix  $U_{TB}$  is just the tribimaximal mixing.

### 11.5 Yukawa interactions $(\bar{\psi}\psi)\phi = ((\mathbf{1}, \mathbf{1}', \mathbf{1}'') \times \mathbf{3}) \times \mathbf{3}$

For the case  $(\bar{\psi}\psi)\phi = ((\mathbf{1}, \mathbf{1}', \mathbf{1}'') \times \mathbf{3}) \times \mathbf{3}$ , we also obtain

$$H_{(\mathbf{1}, \mathbf{3}, \mathbf{3})} = y_0\bar{\psi}_\sigma(\phi\psi)_\sigma + \frac{1}{2}(y_1 + y_2) \left[ \bar{\psi}_\eta(\phi\psi)_\eta + \bar{\psi}_\pi(\phi\psi)_\pi \right] + \frac{i}{2}(y_1 - y_2) \left[ \bar{\psi}_\pi(\phi\psi)_\eta - \bar{\psi}_\eta(\phi\psi)_\pi \right]. \quad (11.5.1)$$

Therefore, when we define the mass matrix  $M$  as

$$H_{(\mathbf{1}, \mathbf{3}, \mathbf{3})} = (\bar{\psi}_\eta \ \bar{\psi}_\sigma \ \bar{\psi}_\pi)M \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad (11.5.2)$$

we obtain

$$M = \begin{pmatrix} \frac{y_1+y_2}{\sqrt{6}}v_1 & -\left(\frac{y_1+y_2}{2}\frac{1}{\sqrt{6}} - i\frac{y_1-y_2}{2}\frac{1}{\sqrt{2}}\right)v_2 & -\left(\frac{y_1+y_2}{2}\frac{1}{\sqrt{6}} + i\frac{y_1-y_2}{2}\frac{1}{\sqrt{2}}\right)v_3 \\ \frac{1}{\sqrt{3}}y_0v_1 & \frac{1}{\sqrt{3}}y_0v_2 & \frac{1}{\sqrt{3}}y_0v_3 \\ i\frac{y_1-y_2}{2}\frac{1}{\sqrt{6}}v_1 & -\left(\frac{y_1+y_2}{2}\frac{1}{\sqrt{2}} + i\frac{y_1-y_2}{2}\frac{1}{\sqrt{6}}\right)v_2 & \left(\frac{y_1+y_2}{2}\frac{1}{\sqrt{2}} - i\frac{y_1-y_2}{2}\frac{1}{\sqrt{6}}\right)v_3 \end{pmatrix}. \quad (11.5.3)$$

If  $y_0 = y_1 = y_2$ , we obtain

$$M = U_{TB}^T D, \quad (11.5.4)$$

where  $U_{TB}$  and  $D$  are defined by (11.4.6) and (11.4.5), respectively.

### 11.6 Yukawa interactions $(\bar{\psi}\psi)\phi = ((\mathbf{1}, \mathbf{1}', \mathbf{1}'') \times (\mathbf{1}, \mathbf{1}', \mathbf{1}'')) \times (\mathbf{1}, \mathbf{1}', \mathbf{1}'')$

For the case  $(\bar{\psi}\psi)\phi = ((\mathbf{1}, \mathbf{1}', \mathbf{1}'') \times (\mathbf{1}, \mathbf{1}', \mathbf{1}'')) \times (\mathbf{1}, \mathbf{1}', \mathbf{1}'')$ , we obtain

$$(\bar{\psi}\psi)\phi = (\mathbf{1} \times \mathbf{1} + \mathbf{1}' \times \mathbf{1}'' + \mathbf{1}'' \times \mathbf{1}') \times \mathbf{1} + (\mathbf{1} \times \mathbf{1}'' + \mathbf{1}'' \times \mathbf{1} + \mathbf{1}' \times \mathbf{1}') \times \mathbf{1}' + (\mathbf{1} \times \mathbf{1}' + \mathbf{1}' \times \mathbf{1} + \mathbf{1}'' \times \mathbf{1}'') \times \mathbf{1}'', \quad (11.6.1)$$

i.e.

$$\begin{aligned} H_{(\mathbf{1}, \mathbf{1}, \mathbf{1})} = & \left[ y_0 \bar{\psi}_\sigma \psi_\sigma + y_1 (\bar{\psi}_\eta \psi_\eta + \bar{\psi}_\pi \psi_\pi) \right] \phi_\sigma + y_2 \bar{\psi}_\sigma (\psi_\pi \phi_\pi + \psi_\eta \phi_\eta) \\ & + y_3 (\bar{\psi}_\pi \phi_\pi + \bar{\psi}_\eta \phi_\eta) \psi_\sigma + y_4 \frac{1}{\sqrt{2}} (\bar{\psi}_\eta \psi_\eta - \bar{\psi}_\pi \psi_\pi) \phi_\eta - y_5 \frac{1}{\sqrt{2}} (\bar{\psi}_\eta \psi_\pi + \bar{\psi}_\pi \psi_\eta) \phi_\pi. \end{aligned} \quad (11.6.2)$$

Here, in (11.6.2), we have assumed the  $\mathbf{1}' \leftrightarrow \mathbf{1}''$  symmetry. Therefore, we obtain the mass matrix on the  $(\psi_\sigma, \psi_\eta, \psi_\pi)$  basis

$$M = \begin{pmatrix} y_0 v_\sigma & y_2 v_\eta & y_2 v_\pi \\ y_3 v_\eta & y_1 v_\sigma + \frac{1}{\sqrt{2}} y_4 v_\eta & -\frac{1}{\sqrt{2}} y_5 v_\pi \\ y_3 v_\pi & -\frac{1}{\sqrt{2}} y_5 v_\pi & y_1 v_\sigma - \frac{1}{\sqrt{2}} y_4 v_\eta \end{pmatrix}. \quad (11.6.3)$$

If we take

$$y_0 = y_1 = y_2 = y_3 = y_4 = y_5 = \frac{1}{\sqrt{3}}, \quad (11.6.4)$$

the mass matrix on the basis  $(\psi_1, \psi_2, \psi_3)$  is given by

$$M = \text{diag}(v_1, v_2, v_3), \quad (11.6.5)$$

where

$$\begin{pmatrix} \psi_\sigma \\ \psi_\eta \\ \psi_\pi \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}. \quad (11.6.6)$$

Note that  $(\psi_1, \psi_2, \psi_3)$  in (11.6.6) is not a triplet of  $A_4$ .

## 12 $S_4$ Symmetry and its Related Matrices

The non-Abelian finite group  $S_4$  is also useful the description of the neutrino mass matrix. In this chapter, we will investigate possible mass matrices under the  $S_4$  invariance.

### 12.1 Irreducible representations of $S_4$

The  $S_4$  symmetry which is the permutation group of four distinct objects has five irreducible representations<sup>15</sup> which are all real. Hereafter, we explicitly denote them as follows:

$$\mathbf{1} = \phi_\sigma, \quad \mathbf{1}' = \phi'_\sigma, \quad \mathbf{2} = \begin{pmatrix} \phi_\pi \\ \phi_\eta \end{pmatrix}, \quad \mathbf{3} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \quad \mathbf{3}' = \begin{pmatrix} \phi'_1 \\ \phi'_2 \\ \phi'_3 \end{pmatrix}. \quad (12.1.1)$$

Then, the products among them are given as follows<sup>16</sup>

$$\begin{aligned} \mathbf{1} \times \mathbf{1} &= \mathbf{1}, & \mathbf{1}' \times \mathbf{1}' &= \mathbf{1}, & \mathbf{1} \times \mathbf{1}' &= \mathbf{1}', \\ \mathbf{2} \times \mathbf{1} &= \mathbf{2}, & \mathbf{2} \times \mathbf{1}' &= \mathbf{2}, & \mathbf{2} \times \mathbf{2} &= (\mathbf{1} + \mathbf{2})_S + \mathbf{1}'_A, \\ \mathbf{3} \times \mathbf{1} &= \mathbf{3}, & \mathbf{3} \times \mathbf{1}' &= \mathbf{3}', & \mathbf{3}' \times \mathbf{1} &= \mathbf{3}', & \mathbf{3}' \times \mathbf{1}' &= \mathbf{3}, \\ \mathbf{3} \times \mathbf{2} &= \mathbf{3} + \mathbf{3}', & \mathbf{3}' \times \mathbf{2}' &= \mathbf{3} + \mathbf{3}', \\ \mathbf{3} \times \mathbf{3} &= (\mathbf{1} + \mathbf{2} + \mathbf{3})_S + \mathbf{3}'_A, & \mathbf{3}' \times \mathbf{3}' &= (\mathbf{1} + \mathbf{2} + \mathbf{3})_S + \mathbf{3}'_A, \\ \mathbf{3} \times \mathbf{3}' &= \mathbf{1}' + \mathbf{2} + \mathbf{3} + \mathbf{3}'. \end{aligned} \quad (12.1.2)$$

These are explicitly given as follows:

$$(\mathbf{2} \times \mathbf{2})_{\mathbf{1}} = \frac{1}{\sqrt{2}}(\phi_\pi\psi_\pi + \phi_\eta\psi_\eta), \quad (\mathbf{2} \times \mathbf{2})_{\mathbf{1}'} = \frac{1}{\sqrt{2}}(\phi_\pi\psi_\eta - \phi_\eta\psi_\pi), \quad (12.1.3)$$

$$(\mathbf{2} \times \mathbf{2})_{\mathbf{2}} = \begin{pmatrix} \frac{1}{\sqrt{2}}(\phi_\pi\psi_\eta + \phi_\eta\psi_\pi) \\ \frac{1}{\sqrt{2}}(\phi_\pi\psi_\pi - \phi_\eta\psi_\eta) \end{pmatrix}, \quad (12.1.4)$$

$$(\mathbf{2} \times \mathbf{3})_{\mathbf{3}} = \begin{pmatrix} \phi_\eta\psi_1 \\ -\frac{1}{2}(\sqrt{3}\phi_\pi + \phi_\eta)\psi_2 \\ \frac{1}{2}(\sqrt{3}\phi_\pi - \phi_\eta)\psi_3 \end{pmatrix}, \quad (\mathbf{2} \times \mathbf{3})_{\mathbf{3}'} = \begin{pmatrix} \phi_\pi\psi_1 \\ \frac{1}{2}(\sqrt{3}\phi_\eta - \phi_\pi)\psi_2 \\ -\frac{1}{2}(\sqrt{3}\phi_\eta + \phi_\pi)\psi_3 \end{pmatrix}, \quad (12.1.5)$$

$$(\mathbf{2} \times \mathbf{3}')_{\mathbf{3}} = \begin{pmatrix} \phi_\pi\psi'_1 \\ \frac{1}{2}(\sqrt{3}\phi_\eta - \phi_\pi)\psi'_2 \\ -\frac{1}{2}(\sqrt{3}\phi_\eta + \phi_\pi)\psi'_3 \end{pmatrix}, \quad (\mathbf{2} \times \mathbf{3}')_{\mathbf{3}'} = \begin{pmatrix} \phi_\eta\psi'_1 \\ -\frac{1}{2}(\sqrt{3}\phi_\pi + \phi_\eta)\psi'_2 \\ \frac{1}{2}(\sqrt{3}\phi_\pi - \phi_\eta)\psi'_3 \end{pmatrix}, \quad (12.1.6)$$

$$(\mathbf{3} \times \mathbf{3})_{\mathbf{1}} = \frac{1}{\sqrt{3}}(\phi_1\psi_1 + \phi_2\psi_2 + \phi_3\psi_3), \quad (\mathbf{3} \times \mathbf{3})_{\mathbf{2}} = \begin{pmatrix} \frac{1}{\sqrt{2}}(-\phi_2\psi_2 + \phi_3\psi_3) \\ \frac{1}{\sqrt{6}}(2\phi_1\psi_1 - \phi_2\psi_2 - \phi_3\psi_3) \end{pmatrix}, \quad (12.1.7)$$

$$(\mathbf{3} \times \mathbf{3})_{\mathbf{3}} = \begin{pmatrix} \frac{1}{\sqrt{2}}(\phi_2\psi_3 + \phi_3\psi_2) \\ \frac{1}{\sqrt{2}}(\phi_3\psi_1 + \phi_1\psi_3) \\ \frac{1}{\sqrt{2}}(\phi_1\psi_2 + \phi_2\psi_1) \end{pmatrix}, \quad (\mathbf{3} \times \mathbf{3})_{\mathbf{3}'} = \begin{pmatrix} \frac{1}{\sqrt{2}}(\phi_2\psi_3 - \phi_3\psi_2) \\ \frac{1}{\sqrt{2}}(\phi_3\psi_1 - \phi_1\psi_3) \\ \frac{1}{\sqrt{2}}(\phi_1\psi_2 - \phi_2\psi_1) \end{pmatrix}, \quad (12.1.8)$$

<sup>15</sup>For a review of  $S_4$ , for example, see C. Hagedorn, M. Lindner and R. N. Mohapatra, JHEP **0606**, 042 (2006).

<sup>16</sup>The phase convention follows Fusaoka's one, which is useful for the investigation of the tribimaximal mixing (9.5.1). The author is much indebted to H. Fusaoka for all the formulae in this section 11.1.

$$(\mathbf{3}' \times \mathbf{3}')_{\mathbf{1}} = \frac{1}{\sqrt{3}}(\phi'_1\psi'_1 + \phi'_2\psi'_2 + \phi'_3\psi'_3), \quad (\mathbf{3}' \times \mathbf{3}')_{\mathbf{2}} = \begin{pmatrix} \frac{1}{\sqrt{2}}(-\phi'_2\psi'_2 + \phi'_3\psi'_3) \\ \frac{1}{\sqrt{6}}(2\phi'_1\psi'_1 - \phi'_2\psi'_2 - \phi'_3\psi'_3) \end{pmatrix}, \quad (12.1.9)$$

$$(\mathbf{3}' \times \mathbf{3}')_{\mathbf{3}} = \begin{pmatrix} \frac{1}{\sqrt{2}}(\phi'_2\psi'_3 + \phi'_3\psi'_2) \\ \frac{1}{\sqrt{2}}(\phi'_3\psi'_1 + \phi'_1\psi'_3) \\ \frac{1}{\sqrt{2}}(\phi'_1\psi'_2 + \phi'_2\psi'_1) \end{pmatrix}, \quad (\mathbf{3}' \times \mathbf{3}')_{\mathbf{3}'} = \begin{pmatrix} \frac{1}{\sqrt{2}}(\phi'_2\psi'_3 - \phi'_3\psi'_2) \\ \frac{1}{\sqrt{2}}(\phi'_3\psi'_1 - \phi'_1\psi'_3) \\ \frac{1}{\sqrt{2}}(\phi'_1\psi'_2 - \phi'_2\psi'_1) \end{pmatrix}, \quad (12.1.10)$$

$$(\mathbf{3} \times \mathbf{3}')_{\mathbf{1}'} = \frac{1}{\sqrt{3}}(\phi_1\psi'_1 + \phi_2\psi'_2 + \phi_3\psi'_3), \quad (\mathbf{3} \times \mathbf{3}')_{\mathbf{2}} = \begin{pmatrix} \frac{1}{\sqrt{6}}(2\phi_1\psi'_1 - \phi_2\psi'_2 - \phi_3\psi'_3) \\ \frac{1}{\sqrt{2}}(\phi_2\psi'_2 - \phi_3\psi'_3) \end{pmatrix}, \quad (12.1.11)$$

$$(\mathbf{3} \times \mathbf{3}')_{\mathbf{3}} = \begin{pmatrix} \frac{1}{\sqrt{2}}(\phi_2\psi'_3 - \phi_3\psi'_2) \\ \frac{1}{\sqrt{2}}(\phi_3\psi'_1 - \phi_1\psi'_3) \\ \frac{1}{\sqrt{2}}(\phi_1\psi'_2 - \phi_2\psi'_1) \end{pmatrix}, \quad (\mathbf{3} \times \mathbf{3}')_{\mathbf{3}'} = \begin{pmatrix} \frac{1}{\sqrt{2}}(\phi_2\psi'_3 + \phi_3\psi'_2) \\ \frac{1}{\sqrt{2}}(\phi_3\psi'_1 + \phi_1\psi'_3) \\ \frac{1}{\sqrt{2}}(\phi_1\psi'_2 + \phi_2\psi'_1) \end{pmatrix}. \quad (12.1.12)$$

## 12.2 Yukawa interactions $(\bar{\psi}\psi)\phi = (\mathbf{3} \times \mathbf{3})_R \times R$

Yukawa interactions  $(\bar{\psi}\psi)\phi = (\mathbf{3} \times \mathbf{3})_R \times R$  are given by

$$H = (\bar{\psi}_1 \ \bar{\psi}_2 \ \bar{\psi}_3) \left[ \frac{y_1}{\sqrt{3}}\phi_\sigma \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + y_2 \begin{pmatrix} \frac{2}{\sqrt{6}}\phi_\eta & 0 & 0 \\ 0 & -\frac{1}{\sqrt{6}}\phi_\eta - \frac{1}{\sqrt{2}}\phi_\pi & 0 \\ 0 & 0 & -\frac{1}{\sqrt{6}}\phi_\eta + \frac{1}{\sqrt{2}}\phi_\pi \end{pmatrix} \right. \\ \left. + \frac{y_3}{\sqrt{2}} \begin{pmatrix} 0 & \phi_3 & \phi_2 \\ \phi_3 & 0 & \phi_1 \\ \phi_2 & \phi_1 & 0 \end{pmatrix} + \frac{y'_3}{\sqrt{2}} \begin{pmatrix} 0 & \phi'_3 & -\phi'_2 \\ -\phi'_3 & 0 & \phi'_1 \\ \phi'_2 & -\phi'_1 & 0 \end{pmatrix} \right] \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad (12.2.1)$$

for  $R = \mathbf{1}, \mathbf{2}, \mathbf{3}$  and  $\mathbf{3}'$ , respectively.

The case of  $(\bar{\psi}\psi)\phi = (\mathbf{3}' \times \mathbf{3}')_R \times R$  is given by the replacement  $(\bar{\psi}_1, \bar{\psi}_2, \bar{\psi}_3) \rightarrow (\bar{\psi}'_1, \bar{\psi}'_2, \bar{\psi}'_3)$  and  $(\psi_1, \psi_2, \psi_3) \rightarrow (\psi'_1, \psi'_2, \psi'_3)$  in Eq.(12.2.1).

On the other hand, the case of  $(\bar{\psi}\psi)\phi = (\mathbf{3} \times \mathbf{3}')_R \times R$  are given by

$$H = (\bar{\psi}_1 \ \bar{\psi}_2 \ \bar{\psi}_3) \left[ \frac{y'_1}{\sqrt{3}}\phi'_\sigma \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + y_2 \begin{pmatrix} \frac{2}{\sqrt{6}}\phi_\eta & 0 & 0 \\ 0 & -\frac{1}{\sqrt{6}}\phi_\eta + \frac{1}{\sqrt{2}}\phi_\pi & 0 \\ 0 & 0 & -\frac{1}{\sqrt{6}}\phi_\eta - \frac{1}{\sqrt{2}}\phi_\pi \end{pmatrix} \right. \\ \left. + \frac{y_3}{\sqrt{2}} \begin{pmatrix} 0 & \phi_3 & -\phi_2 \\ -\phi_3 & 0 & \phi_1 \\ \phi_2 & -\phi_1 & 0 \end{pmatrix} + \frac{y'_3}{\sqrt{2}} \begin{pmatrix} 0 & \phi'_3 & \phi'_2 \\ \phi'_3 & 0 & \phi'_1 \\ \phi'_2 & \phi'_1 & 0 \end{pmatrix} \right] \begin{pmatrix} \psi'_1 \\ \psi'_2 \\ \psi'_3 \end{pmatrix}, \quad (12.2.2)$$

for  $R = \mathbf{1}', \mathbf{2}, \mathbf{3}$  and  $\mathbf{3}'$ , respectively.

## 12.3 Yukawa interactions $(\bar{\psi}\psi)\phi = (\mathbf{3} \times (\mathbf{1} + \mathbf{2}))_R \times R$

Yukawa interactions  $(\bar{\psi}\psi)\phi = (\mathbf{3} \times (\mathbf{1} + \mathbf{2}))_R \times R$  are given by

$$H = \frac{y_1}{\sqrt{3}}(\bar{\psi}_1 \ \bar{\psi}_2 \ \bar{\psi}_3) \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \psi_\sigma$$

$$+(\bar{\psi}_1 \ \bar{\psi}_2 \ \bar{\psi}_3) \left[ y_2 \begin{pmatrix} 0 & \frac{2}{\sqrt{6}}\phi_1 \\ -\frac{1}{\sqrt{2}}\phi_2 & -\frac{1}{\sqrt{6}}\phi_2 \\ \frac{1}{\sqrt{2}}\phi_3 & -\frac{1}{\sqrt{6}}\phi_3 \end{pmatrix} + y'_2 \begin{pmatrix} \frac{2}{\sqrt{6}}\phi'_1 & 0 \\ -\frac{1}{\sqrt{6}}\phi'_2 & \frac{1}{\sqrt{2}}\phi'_2 \\ -\frac{1}{\sqrt{6}}\phi'_3 & -\frac{1}{\sqrt{2}}\phi'_3 \end{pmatrix} \right] \begin{pmatrix} \psi_\pi \\ \psi_\eta \end{pmatrix}. \quad (12.3.1)$$

The case of  $(\bar{\psi}\psi)\phi = (\mathbf{3}' \times (\mathbf{1} + \mathbf{2}))_R \times R$  is given by the replacement  $\psi_i \rightarrow \psi'_i$  and  $\phi_i \leftrightarrow \phi'_i$  in Eq.(12.3.1).

## 13 CKM Quark Mixing Matrix

As an example of flavor mixing matrices, it is useful to summarize formulae related to the quark mixing matrix. Especially, a review of  $CP$ -violating phase conventions of quark mixing matrix will be useful.

### 13.1 Definition of the CKM matrix

The quark mixing matrix (the Cabibbo-Kobayashi-Maskawa<sup>17</sup> (CKM) mixing matrix)  $V$  is defined by

$$V = U_{uL}^\dagger U_{dL}, \quad (13.1.1)$$

where mass matrices of the up- and down-quarks  $M_u$  and  $M_d$  are defined by

$$H_{mass} = \bar{u}_L M_u u_R + \bar{d}_L M_d d_R + h.c., \quad (13.1.2)$$

and those are diagonalized as

$$\begin{aligned} U_{uL}^\dagger M_u U_{uR} &= D_u \equiv \text{diag}(m_u, m_c, m_t), \\ U_{dL}^\dagger M_d U_{dR} &= D_d \equiv \text{diag}(m_d, m_s, m_b). \end{aligned} \quad (13.1.3)$$

If  $M_u$  and  $M_d$  are given, the mixing matrices  $U_{uL}$  and  $U_{dL}$  are obtained from the diagonalization of  $M_u M_u^\dagger$  and  $M_d M_d^\dagger$ :

$$U_{uL}^\dagger M_u M_u^\dagger U_{uL} = D_u^2, \quad U_{dL}^\dagger M_d M_d^\dagger U_{dL} = D_d^2. \quad (13.1.4)$$

### 13.2 Number of independent parameters

Generally, an  $n \times n$  mass matrix  $M_q$  has  $2n^2$  parameters. Since  $M_q M_q^\dagger$  is a Hermitian matrix, it has  $n^2$  parameters:

$$N(M_q) = 2n^2 = N(M_q M_q^\dagger) + N(M_q^\dagger M_q) = n^2 + n^2. \quad (13.2.1)$$

As seen in Eq.(13.1.4), the matrix  $M_q M_q^\dagger$  with  $n^2$  parameters determine the  $n$  eigenvalues  $D_q$  and the mixing matrix  $U_{qL}$ . Although an  $n \times n$  unitary matrix generally has  $n^2$  parameters, in the present case, the mixing matrix  $U_{qL}$  has  $n(n-1)$  parameters, because Eq.(13.1.4) is invariant under the transformation

$$U_{qL} \rightarrow U'_{qL} = U_{qL} P(\delta_{qL}), \quad (13.2.2)$$

where

$$P(\delta) = \text{diag}(e^{i\delta_1}, e^{i\delta_2}, e^{i\delta_3}), \quad (13.2.3)$$

so that  $n$  parameters are free (unobservable).

Although the CKM matrix  $V$  is composed of  $U_{uL}$  with  $n(n-1)$  parameters and  $U_{dL}$  with  $n(n-1)$  parameters, of the  $2n(n-1)$  parameters,  $n^2$  parameters are reduced because of the  $n^2$

<sup>17</sup>N. Cabibbo, Phys. Rev. Lett. **10**, 531 (1963); M. Kobayashi and T. Maskawa, Prog. Theor. Phys. **49**, 652 (1973).

unitary conditions for  $V$ . Furthermore, of the  $n + n$  phases in  $P(\delta_{uL})$  and  $P(\delta_{dL})$ , one is not free (cannot be absorbed into  $u_L$  and  $d_L$ ), so that a number of the independent parameters of  $V$  is

$$N(V) = N(U_{uL}) + N(U_{dL}) - N(\text{unitary cond.}) + 1 = n(n-1) + n(n-1) - n^2 + 1 = (n-1)^2. \quad (13.2.4)$$

For  $n = 3$ , we obtain  $N(V) = 4$ . Since an  $n \times n$  orthogonal matrix has  $n(n-1)/2$  parameters (rotation angles), a number of the  $CP$  violating parameters in  $V$  is

$$N_{CP}(V) = (n-1)^2 - \frac{1}{2}n(n-1) = \frac{1}{2}n(n-1)(n-2). \quad (13.2.5)$$

For  $n = 3$ , we obtain  $N_{CP}(V) = 1$ , i.e. in a three family model, the number of  $CP$  violating phases is only one.

### 13.3 Expressions of the CKM matrix

Since the CKM matrix has a freedom of “rephasing”

$$V \rightarrow P^\dagger(\delta_u)VP(\delta_d), \quad (13.3.1)$$

various expressions of  $V$  are possible as stated in this section.

#### 13.3.1 Standard and Original Kobayashi-Maskawa expressions

Usually, the following phase convention<sup>18</sup> is adopted as the standard expression of  $V$ :

$$\begin{aligned} V_{SD} &= R_1(\theta_{23})P_3(\delta)R_2(\theta_{13})P_3^\dagger(\delta)R_3(\theta_{12}) \\ &= \begin{pmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - s_{23}c_{12}s_{13}e^{i\delta} & c_{23}c_{12} - s_{23}s_{12}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{23}s_{12} - c_{23}c_{12}s_{13}e^{i\delta} & -s_{23}c_{12} - c_{23}s_{12}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}. \end{aligned} \quad (13.3.2)$$

On the other hand the following expression is known as the original Kobayashi-Maskawa matrix phase convention<sup>19</sup>:

$$\begin{aligned} V_{KM} &= R_1^T(\theta_2)P_3(\delta_{KM} + \pi)R_3(\theta_1)R_1(\theta_3) \\ &= \begin{pmatrix} c_1 & -s_1c_3 & -s_1s_3 \\ s_1c_2 & c_1c_2c_3 - s_2s_3e^{i\delta_{KM}} & c_1c_2s_3 + s_2c_3e^{i\delta_{KM}} \\ s_1s_2 & c_1s_2c_3 + c_2s_3e^{i\delta_{KM}} & c_1s_2s_3 - c_2c_3e^{i\delta_{KM}} \end{pmatrix}. \end{aligned} \quad (13.3.3)$$

Here  $R_n$  ( $n = 1, 2, 3$ ) and  $P_3$  are defined by

$$R_1(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix}, \quad R_2(\theta) = \begin{pmatrix} c & 0 & s \\ 0 & 1 & 0 \\ -s & 0 & c \end{pmatrix}, \quad R_3(\theta) = \begin{pmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (13.3.4)$$

$$P_3(\delta) = \text{diag}(1, 1, e^{i\delta}), \quad (13.3.5)$$

$s = \sin \theta$  and  $c = \cos \theta$ .

<sup>18</sup>L. -L. Chau and W. -Y. Keung, Phys. Rev. Lett. **53**, 1802 (1984); H. Fritzsch, Phys. Rev. **D32**, 3058 (1985).

<sup>19</sup>M. Kobayashi and T. Maskawa, Prog. Theor. Phys. **49**, 652 (1973).

### 13.3.2 General expressions of the CKM matrix

There are, in general, 9 independent phase conventions<sup>20</sup> of the CKM matrix. We define the expressions of the CKM matrix  $V$  as

$$V = V(i, k) \equiv R_i^T P_j R_j R_k \quad (i \neq j \neq k), \quad (13.3.6)$$

where  $R_n$  are defined by Eqs.(13.3.5), and

$$\begin{aligned} P_1 &= \text{diag}(e^{i\delta}, 1, 1), \\ P_2 &= \text{diag}(1, e^{i\delta}, 1), \\ P_3 &= \text{diag}(1, 1, e^{i\delta}). \end{aligned} \quad (13.3.7)$$

The expressions  $V(1, 3)$ ,  $V(1, 1)$  and  $V(3, 3)$  correspond to the standard, original KM and Fritzsche-Xing<sup>21</sup> phase conventions, respectively. (Note that, in the standard expression defined by Eq.(13.3.2), the matrices  $P_3(\delta)^\dagger$  and  $R_3(\theta_{12})$  are commutable, so that the phase matrix  $P_3^\dagger(\delta)$  in  $V_{SD} = R_1(\theta_{23})P_3(\delta)R_2(\theta_{13})P_3^\dagger(\delta)R_3(\theta_{12}) = R_1(\theta_{23})P_3(\delta)R_2(\theta_{13})R_3(\theta_{12})P_3^\dagger(\delta)$  can be absorbed into the down-quark field  $d_L$ .)

For relations of those phase conventions to the rephasing invariant quantity  $J$ , see Sec.12.4.

### 13.3.3 Wolfenstein parameterization

The CKM matrix  $V$  can approximately be expressed by

$$V = \begin{pmatrix} 1 - \frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4), \quad (13.3.8)$$

where

$$\begin{aligned} \lambda \equiv s_{12} &= \frac{|V_{us}|}{\sqrt{|V_{ud}|^2 + |V_{us}|^2}}, \quad A\lambda^2 \equiv s_{23} = \lambda \left| \frac{V_{cb}}{V_{us}} \right|, \\ A\lambda^3(\rho + i\eta) \equiv V_{ub}^* &= s_{13}e^{i\delta} = \frac{A\lambda^3(\bar{\rho} + i\bar{\eta})\sqrt{1 - A^2\lambda^4}}{\sqrt{1 - \lambda}[1 - A^2\lambda^4(\bar{\rho} + i\bar{\eta})]}, \end{aligned} \quad (13.3.9)$$

$$\bar{\rho} + i\bar{\eta} = -\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}, \quad (13.3.10)$$

and

$$\bar{\rho} = \rho \left( 1 - \frac{1}{2}\lambda^2 \right) + \dots \quad (13.3.11)$$

The expression (13.3.8) is known as the Wolfenstein parameterization<sup>22</sup>. The expression (13.3.8) is useful for analytical investigation of the CKM matrix. However, for numerical investigation, the expression (13.3.2) without an approximation is useful together with the relations

$$s_{12} = \frac{|V_{us}|}{\sqrt{1 - |V_{ub}|^2}}, \quad s_{22} = \frac{|V_{cb}|}{\sqrt{1 - |V_{ub}|^2}}, \quad s_{13} = |V_{ub}|. \quad (13.3.12)$$

<sup>20</sup>H. Fritzsch and Z. -z. Xing, Phys. Rev. **D57**, 594 (1998).

<sup>21</sup>H. Fritzsch and Z. -z. Xing, Phys. Lett. **B413**, 396 (1997).

<sup>22</sup>L. Wolfenstein, Phys. Rev. Lett. **51**, 1945 (1983).

### 13.3.4 Other expression

If we are interested only in the magnitudes of the CKM matrix elements  $|V_{ij}|$ , the following expression<sup>23</sup> is also useful:

$$|V_{ij}| = \begin{pmatrix} \sqrt{1 - \lambda^2 - \sigma^2} & \lambda & \sigma \\ \sqrt{\lambda^2 + \omega} & \sqrt{1 - \lambda^2 - \rho^2 - \omega} & \rho \\ \sqrt{\sigma^2 - \omega} & \sqrt{\rho^2 + \omega} & \sqrt{1 - \rho^2 - \sigma^2} \end{pmatrix}, \quad (13.3.13)$$

where

$$\lambda = |V_{us}|, \quad \rho = |V_{cb}|, \quad \sigma = |V_{ub}|, \quad (13.3.14)$$

$$\omega = |V_{cd}|^2 - |V_{us}|^2 = |V_{ts}|^2 - |V_{cb}|^2 = |V_{ub}|^2 - |V_{td}|^2, \quad (13.3.15)$$

or

$$V = \begin{pmatrix} \sqrt{1 - \lambda^2 - \sigma^2} & \lambda & \sigma e^{-i\delta} \\ -\sqrt{\lambda^2 + \omega} e^{i\phi_{21}} & \sqrt{1 - \lambda^2 - \rho^2 - \omega} e^{i\phi_{22}} & \rho \\ -\sqrt{\sigma^2 - \omega} e^{i\phi_{31}} & \sqrt{\rho^2 + \omega} e^{i\phi_{32}} & \sqrt{1 - \rho^2 - \sigma^2} \end{pmatrix}, \quad (13.3.16)$$

where

$$\begin{aligned} \tan \phi_{21} &\simeq \pm(\rho\sigma/\lambda) \sin \delta \simeq 0, \\ \tan \phi_{22} &\simeq \pm\lambda\rho\sigma \sin \delta \simeq 0, \end{aligned} \quad (13.3.17)$$

$$\tan \phi_{32} \simeq \pm(\lambda\sigma/\rho) \sin \delta \simeq 0,$$

$$\cos \delta \simeq \frac{\lambda^2\rho^2 + \omega}{2\lambda\rho\sigma}. \quad (13.3.18)$$

## 13.4 Rephasing invariant quantity $J$

The following quantity<sup>24</sup>  $J$  is invariant under the rephasing transformation (13.3.1):

$$\text{Im}(V_{il}V_{jm}V_{jm}^*V_{jl}^*) = J \sum_{k,n} \varepsilon_{ijk} \varepsilon_{lmn}, \quad (13.4.1)$$

i.e.

$$J = \text{Im}(V_{ud}V_{cs}V_{us}^*V_{cd}^*) = \text{Im}(V_{cb}V_{us}V_{cs}^*V_{ub}^*) = \dots \quad (13.4.2)$$

The quantity  $J$  is explicitly given by

$$J = s_{12}c_{12}s_{23}c_{23}s_{13}c_{13}^2 \sin \delta, \quad (13.4.3)$$

for the standard phase convention (13.3.2), and by

$$J = s_1^2 c_1 s_2 c_2 s_3 c_3 \sin \delta, \quad (13.4.4)$$

for the original KM phase convention (13.3.3). For the general expression (13.3.6), the rephasing invariant  $J$  is given by the form<sup>25</sup>

$$J = \frac{|V_{i1}||V_{i2}||V_{i3}||V_{1k}||V_{2k}||V_{3k}|}{(1 - |V_{ik}|^2)|V_{ik}|} \sin \delta. \quad (13.4.5)$$

<sup>23</sup>Y. Koide, Mod. Phys. Lett. **A7**, 1691 (1992); G. Bélanger, C. Hamzaoui and Y. Koide, Phys. Rev. **D45**, 4186 (1992).

<sup>24</sup>C. Jarlskog, Z. Phys. **C29**, 491 (1985); Phys. Rev. Lett. **55**, 1039 (1985).

<sup>25</sup>Y. Koide, Phys. Rev. **D73**, 073002 (2006)

The quantity  $J$  can also be given by the form<sup>26</sup> without the phase parameter:

$$J^2 = |V_{ub}^2|V_{cb}|^2|V_{ud}|^2|V_{cd}|^2 - \frac{1}{4} \left( |V_{cs}|^2|V_{us}|^2 - |V_{cd}|^2|V_{ud}|^2 - |V_{ub}|^2|V_{cb}|^2 \right)^2, \quad (13.4.6)$$

or

$$J^2 = \lambda^2 \rho^2 \sigma^2 - \frac{1}{4} \left[ \lambda^2 \rho^2 + (\lambda^2 + \rho^2) \sigma^2 \right]^2 - \frac{1}{2} \left[ \lambda^2 \rho^2 (1 + \sigma^2) - (\lambda^2 + \rho^2) \sigma^2 (1 - \sigma^2) \right] \omega - \frac{1}{4} (1 - \sigma^2)^2 \omega^2, \quad (13.4.7)$$

for the expression (13.3.13).

### 13.5 Unitary triangle

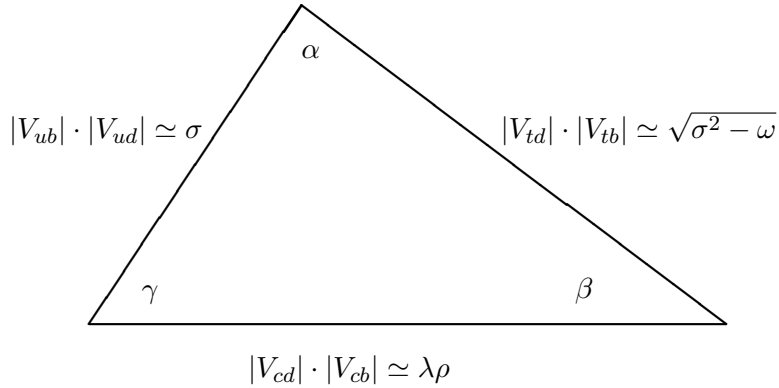
The CKM matrix  $V$  satisfies a unitary conditions

$$\sum_k V_{ki}^* V_{kj} = \delta_{ij}. \quad (13.5.1)$$

Especially, a triangle (Fig.12.5.1) which is illustrated from the relation

$$V_{ud}^* V_{ub} + V_{cd}^* V_{cb} + V_{td}^* V_{tb} = 0, \quad (13.5.2)$$

is called as the “unitary triangle”, and it is very useful for experimental studies of the CKM matrix. The area of the triangle gives  $J/2$ .



**Fig. 13.5.1 Unitary triangle**

In Fig. 13.5.1, the angles  $\alpha$ ,  $\beta$  and  $\gamma$  are defined by

$$\alpha \equiv \phi_2 = \text{Arg} \left[ -\frac{V_{31} V_{33}^*}{V_{11} V_{13}^*} \right], \quad \beta \equiv \phi_1 = \text{Arg} \left[ -\frac{V_{21} V_{23}^*}{V_{31} V_{33}^*} \right], \quad \gamma \equiv \phi_3 = \text{Arg} \left[ -\frac{V_{11} V_{13}^*}{V_{21} V_{23}^*} \right], \quad (13.5.3)$$

respectively. The angles  $(\phi_1, \phi_2, \phi_3) \equiv (\beta, \alpha, \gamma)$  on the unitary triangle are given by the sine rule

$$\frac{r_1}{\sin \phi_1} = \frac{r_2}{\sin \phi_2} = \frac{r_3}{\sin \phi_3} = 2R, \quad (13.5.4)$$

<sup>26</sup>C. Hamzaoui, Phys. Rev. Lett. **61**, 35 (1988); G. C. Branco and L. Lavoura, Phys. Lett. **B208**, 123 (1985).

where  $R$  is the radius of the circumscribed circle of the unitary triangle, and  $r_i$  are defined by

$$r_1 = |V_{13}||V_{11}|, \quad r_2 = |V_{23}||V_{21}|, \quad r_3 = |V_{33}||V_{31}|. \quad (13.5.5)$$

Then, the quantity  $J$  is written as follows:

$$J = r_m r_n \sin \phi_\ell = \frac{1}{2R} r_\ell r_m r_n = \frac{1}{2R} |V_{11}||V_{21}||V_{31}||V_{13}||V_{23}||V_{33}|, \quad (13.5.6)$$

where  $(\ell, m, n)$  is a cyclic permutation of  $(1, 2, 3)$ . From Eqs. (13.4.5), (13.5.4) and (13.5.6), the angles  $\phi_\ell$  are given by the formula<sup>27</sup>

$$\sin \phi_\ell = \frac{|V_{i1}||V_{i2}||V_{i3}||V_{1k}||V_{2k}||V_{3k}| \sin \delta}{|V_{m1}||V_{m3}||V_{n1}||V_{n3}|(1 - |V_{ik}|^2)|V_{ik}|}. \quad (13.5.7)$$

Note that, of the three sides in the expression  $V(i, k)$ , only one side  $r_i$  is always independent of the phase parameter  $\delta$ .

### 13.6 Relation of $J$ to quark mass matrices

The rephasing invariant  $J$  is expressed by the up- and down-quark mass matrices,  $M_u$  and  $M_d$ , as follows:

$$\det(M_u M_d - M_d M_u) = 2i \kappa_u \kappa_d J, \quad (13.6.1)$$

for Hermitian matrices  $M_u$  and  $M_d$ , where

$$\begin{aligned} \kappa_u &= (m_t - m_c)(m_t - m_u)(m_c - m_u), \\ \kappa_d &= (m_b - m_s)(m_b - m_d)(m_s - m_d). \end{aligned} \quad (13.6.2)$$

The formula (13.6.1) can be derived by using the formula (4.1.3) for the traceless matrix  $A_0 = M_u M_d - M_d M_u$ . If the quark mass matrices are not Hermitian, we can also obtain a relation

$$\det(H_u H_d - H_d H_u) = 2i \kappa_u^2 \kappa_d^2 J, \quad (13.6.3)$$

where  $H_q = M_q M_q^\dagger$  ( $q = u, d$ ) and

$$\begin{aligned} \kappa_u^2 &= (m_t^2 - m_c^2)(m_t^2 - m_u^2)(m_c^2 - m_u^2), \\ \kappa_d^2 &= (m_b^2 - m_s^2)(m_b^2 - m_d^2)(m_s^2 - m_d^2). \end{aligned} \quad (13.6.4)$$

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<sup>27</sup>Y. Koide, Phys. Rev. **D73**, 073002 (2006)